Atlas of Coordinate Charts on the de Sitter Spacetime

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Abstract

The de Sitter manifold admits a wide variety of interesting coordinatizations. The 'atlas' is a compilation of the coordinate charts referenced throughout the literature, and is presented in the form of tables, the starting point being the embedding in a higher-dimensional Minkowski spacetime. The metric tensor and the references where the coordinate frame is discussed or used in applications are noted. Additional information is given for the entries with significant use: a convenient tetrad and the form taken by the Killing vectors in the respective coordinate frame.

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1 Introduction

This work is an attempt to bring together all the known coordinate frames that have been utilised or mentioned in the literature for de Sitter spacetime. There have been a number of rather thorough reviews in this sense, the most important of which is the one made by Eriksen and Grøn [14], but also in part by Schmidt [13], Bičák and Krtouš [15] or Spradlin, Strominger and Volovich [20].

Here, emphasis is put on the embedding in a higher-dimensional Minkowskian spacetime and the form of the line element as it results from this embedding. The de Sitter manifold $d\mathbb{S}^{1,n}$ (Lorenzian manifold if dimension $n+1$) can be envisioned as a hyperboloid embedded in an $n+2$- dimensional flat Minkowski maifold $\mathbb{M}^{1,n+1}$, acording to the constraint

$$\eta_{AB} Z^A Z^B = -\frac{1}{\omega^2}$$

where η_{AB} is the metric on $\mathbb{M}^{1,n+1}$, A, B are indices that run from 0 to $n+1$, $\{Z^A\}$ is the standard cartesian coordinate chart on $\mathbb{M}^{1,n+1}$ and ω is the Hubble constant. In this work, we consider only the case with three spatial dimensions ($n=3$), where the results have more familiar forms, but they can be generalised to arbitrary n.

The $d\mathbb{S}$ is parametrized by coordinate charts generically denoted by $\{x^\mu\}$, where μ runs from 0 to $n=3$. Then, the metric on $d\mathbb{S}$ that is inherited through the embedding $Z^A(x^\mu)$ is

$$g_{\mu\nu} = \eta_{AB} \frac{\partial Z^A}{\partial x^\mu} \frac{\partial Z^B}{\partial x^\nu}$$

At a coordinate change, the metric transforms just like a tensor:

$$g'_{\alpha\beta} = g_{\mu\nu} \frac{\partial x^\mu}{\partial x^\alpha} \frac{\partial x^\nu}{\partial x^\beta}$$

The de Sitter manifold, just as the Minkowski manifold is maximally symmetric (it has the maximum amount of Killing vectors- 10 for the total 4 dimensions). The Killing vectors components on de Sitter can be expressed in terms of the embedding, being inherited also from the embedding space:

$$k^\mu_{AB} = g^{\mu\nu} \eta_{AC} \eta_{BD} \left(Z^A \frac{\partial Z^B}{\partial x^\nu} - Z^B \frac{\partial Z^A}{\partial x^\nu} \right)$$

Therefore, this is a simple way for computing their forms in various charts, not being necessary to get them by solving the Killing equations:

$$\nabla_\mu k^a_\nu + \nabla_\nu k^a_\mu = 0$$

Also, for a coordinate chart transformation, their components transform as componets of a vector:

$$k^{\alpha}_{AB} = k^{\mu}_{AB} \frac{\partial x^{\mu}}{\partial x^{\alpha}}$$

which is a convenient relation for computing the Killing vector components between closely related charts, or those that belong to the same class of coordinates.

We do not stress too much the explicit transformations between coordinates, as in [14]. The transformations between the main classes of coordinates are given there, and they can be combined with the ones at the beginning of each section of the present work, in order to obtain the coordinate transformation from any one chart to another.

The work is formatted as tables in order to provide the most clear overview possible. It is intended to be used as a quick reference and guide to the various coordinatisations of the de Sitter manifold, each used in order to express different properties, or to emphasise different connections to other spacetimes.

As a convention, all coordinates that have a physical meaning of spatial, or temporal coordinate are expressed in terms of length (dimension of $\frac{1}{\omega}$). Sometimes, in the literature, they are scaled in order to be adimensional: $r_{sc} = \omega r$. We do not differentiate between scaled adimensional coordinates and those which have dimension of length (we consider the chart to be the "same" one). In some references, the constant ω is dropped altogether from the line element.

In other cases, coordinates are shifted like $r_{sh} = r + a$ usually with a factor of $a = -\frac{\pi}{2\omega}$, the domain being shifted from $r \in \left(0, \frac{\pi}{\omega}\right)$ to $r_{sh} \in \left(-\frac{\pi}{2\omega}, \frac{\pi}{2\omega}\right)$. This is applied to both temporal coordinates, and radial coordinates. It occurs when for example in the line element sin can be equivalently replaced with cos. Then $\sin(\omega r) = \cos(\omega r_{sh})$ and $\cos(\omega r) = -\sin(\omega r_{sh})$. "Shifted" coordinates defined in this way are denoted by a prime superscript, and are presented only if they have been identified at least once in the cited literature.

There is also when the coordinates are "rotated" in the sense of the coodinate used being pure imaginary: $r_{rot} = ir$. This has been done even by de Sitter himself, as Schmidt [13] carefully points out. Then, $\cos(\omega r) = \cosh(\omega r_{rot})$, $\cosh(\omega r) = \cos(\omega r_{rot})$ and $\sin(\omega r) = -i \sinh(\omega r_{rot})$, $\sinh(\omega r) = -i \sin(\omega r_{rot})$. We do not consider here coordinates with purely imaginary domain, since they are always reducible to ones with real domain. However, there is one instance of a coordinate with complex domain referenced in Table 35.

While these considerations might seem trivial, the lack of a standard convention, or even nomenclature and notation for the coordinates often leads to confusion.

Most of the charts do not cover the manifold in its entirety. In fact, *sensu stricto* there is no global chart on de Siiter space, since the spherically symmetric charts, of the type $\{t, r, \theta, \phi\}$ have two trivial coordinate symmetries at the points corresponding to $\theta = 0, \phi = 2\pi$, just as there is no global chart on the sphere manifold $\mathbb{S}^2$, the chart $\{\theta, \phi\}$ covering everything, except the two poles. We will use the term "global chart" in the relaxed sense (as most of the physics literature tacitly assumes), meaning a chart with possibly trivial coordinate singularities. However, it can happen that such a chart is global, but not smoothly global (a singular surface is present).

Charts that introduce additonal parameters, such as Bičák and Krtouš's "accelerated coordinates" [15] are also not considered.

The charts are grouped in an synthetic way, rather than being classified mathematically. For charts with spherical symmetry (except the ones with at least a null coordinate), the version in spherical coordinates is presented head-to-head with the one in cartesian (or more correctly called "pseudo-cartesian") coordinates, even if that one does not appear in the literature. The transformation is the usual one:

$$
\begin{aligned}
x &= r \sin\theta \cos\phi \\
y &= r \sin\theta \sin\phi \\
z &= r \cos\theta
\end{aligned}
$$

The spherical line element on the sphere manifold $\mathbb{S}^2$ is denoted $d\Omega_2^2 = d\theta^2 + \sin^2\theta d\phi^2$, while the one on $\mathbb{H}^{1,1}$ is $dH_2^2 = d\Theta^2 + \sinh^2\Theta d\phi^2$.

The notation used for the coordinates in the embedding space is Z^A, where the index $A = 0..5$. For the coordinates on the de Sitter hyperboloid, the notation is x^μ, where $\mu = 0..4$. The temporal index is denoted by 0 and the spatial ones by $i, j = 1..3$. Note that while $x^0 = x_0$, $x^i = -x_i$. Therefore $x^\mu x_\mu = t^2 - x^{i2}$ and $x^\mu \partial_\mu = t\partial_0 - x^i \partial_i$.

For the various charts, every distinct coordinate is given a different symbol, sometimes with a subscript or an overline. Unfortunately, since the notation across sources is most always inconsistent, we tried to both keep notation used in some works, but also refrain from using the same notations for different coordinates.

Every quantity is expressed using these symbols, except in the case of the Killing vectors, where they are referenced generically as $\{t, x^i\}$ and their derivatives as ∂_0 and ∂_i respectively.

We start with 4 families of rather special charts- that are diagonal when expressed in spherical coordinates. 3 of them are the FLRW families, and one is the static family. Of course, any one coordinate can be transformed arbitrarily, giving rise to an infinite number of coordinate charts pertaining to each family. However, they

are almost exclusively found in the literature in forms that have the line element of the metric expressed as a FLRW line element in one of its 6 forms- 3 proper, or 3 conformal [46] (or similar to that- for the static family). These common charts, presented in Sections 2 and 3 can be neatly arranged in the following table, according to the form taken by the line element expressed in spherical coordinates.

Table 1: **The 4 families with spherical diagonal line elements**

<table>
<tr><td colspan="3" rowspan="3">$b = b(t,r)$
$a = a(t,r)$
$s_i = s_i(t,r)$
$\Omega = \Omega(t,r)$</td><td colspan="4">proper forms</td><td colspan="4">conformal forms</td></tr>
<tr><td colspan="2">standard</td><td>alternate</td><td colspan="2">isotropic</td><td>alternate</td><td colspan="2">standard</td></tr>
<tr><td colspan="2">$b^2 \cdot dt^2 - a^2 \cdot$
$\cdot (dr^2 + s_1^2 \cdot d\Omega_2^2)$</td><td>$b^2 \cdot dt^2 - a^2$
$\cdot (s_2^2 \cdot dr^2 +$
$+ r^2 d\Omega_2^2)$</td><td>$b^2 \cdot dt^2 - a^2$
$\cdot s_3^2 \cdot (dr^2 +$
$+ r^2 d\Omega_2^2)$</td><td>$\Omega^2 \cdot [dt^2 -$
$- s_3^2 \cdot (dr^2 +$
$+ r^2 d\Omega_2^2)]$</td><td>$\Omega^2 \cdot [dt^2 -$
$- s_2^2 \cdot dr^2 -$
$- r^2 d\Omega_2^2)$</td><td colspan="2">$\Omega^2 \cdot (dt^2 -$
$- dr^2 - s_1^2 \cdot d\Omega_2^2)$</td></tr>
<tr><td rowspan="5">Isotropic
$s_i = s_i(r)$</td><td rowspan="4">FLRW
$b = 1$
$a = a(t)$
$\Omega = \Omega(t)$</td><td rowspan="2">k=+1</td><td rowspan="2">$\{t_{+1}, r_{+1}\}$</td><td rowspan="2">$\{t_{+1}, r'_{+1}\}$</td><td rowspan="2">$\{t_{+1}, \bar{r}_{+1}\}$</td><td rowspan="2">$\{t_{+1}, \rho_{+1}\}$</td><td>$\{\eta_{+1}, \rho_{+1}\}$</td><td>$\{\eta_{+1}, \bar{r}_{+1}\}$</td><td>$\{\eta_{+1}, r_{+1}\}$</td><td>$\{\eta_{+1}, r'_{+1}\}$</td></tr>
<tr><td>$\{\eta'_{+1}, \rho_{+1}\}$</td><td>$\{\eta'_{+1}, \bar{r}_{+1}\}$</td><td>$\{\eta'_{+1}, r_{+1}\}$</td><td>$\{\eta'_{+1}, r'_{+1}\}$</td></tr>
<tr><td>k=0</td><td colspan="4">$\{t, r\}$</td><td colspan="4">$\{\eta, r\}$</td></tr>
<tr><td>k=-1</td><td colspan="2">$\{t_{-1}, r_{-1}\}$</td><td>$\{t_{-1}, \bar{r}_{-1}\}$</td><td>$\{t_{-1}, \rho_{-1}\}$</td><td>$\{t_{-1}, \rho_{-1}\}$</td><td>$\{t_{-1}, \bar{r}_{-1}\}$</td><td colspan="2">$\{t_{-1}, r_{-1}\}$</td></tr>
<tr><td colspan="2">static $b = b(r)$
$a = 1,\ \Omega = \Omega(r)$</td><td>$\{t_s, \bar{r}_F\}$</td><td>$\{t_s, \bar{r}'_F\}$</td><td>$\{t_s, r_s\}$</td><td colspan="2">$\{t_s, \rho_s\}$</td><td>$\{t_s, \bar{r}_s\}$</td><td colspan="2">$\{t_s, r^*\}$</td></tr>
</table>

Further, we mean by 'natural charts'- the charts for which there is an embedding

$$Z^\mu = \frac{x^\mu}{f(x_\nu x^\nu)}$$

All the coordinates that make up the special four families of charts with diagonal line elements in spherical coordinates and the natural charts are defined in the context of the special properties that these charts have. But in fact, coordinates from any chart can be combined with any other, provided no two coordinates have an exclusive dependence on eachother. This is what we mean by the made-up notion of "hybrid" coordinates. Because the metric in these situations is usually more complicated (it can containing even $dtdx$ terms), charts of this type are not usually taken into consideration in applications.

After that, some anisotropic charts are presented, and after that charts with null coordinates, of Eddington-Finkelstein, or Kruskal type- named in reference to their analogues defined on the Schwarschild geometry.

Note: This work is not intended for publication. It emerged as a result of a PhD report of the author, under the supervison of prof. dr. Ion I. Cotăescu, to whom the author is grateful for guidance, encouragement and useful suggestions.

2 FLRW charts

Table 2: **FLRW coordinates on de Sitter spacetime**

k=+1 FLRW radii [14, rel.(5.2),(5.7)]

r_{+1}	$r_{+1} = \frac{1}{\omega}\arcsin(\omega\bar{r}_{+1}) = \frac{2}{\omega}\arctan\frac{\omega\rho_{+1}}{2}$
	$dr_{+1} = \frac{d\bar{r}_{+1}}{\sqrt{1-\omega^2\bar{r}_{+1}^2}} = \frac{d\rho_{+1}}{1+\frac{\omega^2\rho_{+1}^2}{4}}$
$\bar{r}_{+1}$	$\bar{r}_{+1} = \frac{1}{\omega}\sin(\omega r_{+1}) = \frac{\rho_{+1}}{1+\frac{\omega^2\rho_{+1}^2}{4}}$
	$d\bar{r}_{+1} = \cos(\omega r_{+1})dr_{+1} = 4\frac{1-\frac{\omega^2\rho_{+1}^2}{4}}{\left(1+\frac{\omega^2\rho_{+1}^2}{4}\right)^2}d\rho_{+1}$
ρ_{+1}	$\rho_{+1} = \frac{2}{\omega}\tan\frac{\omega r_{+1}}{2} = \frac{2}{\omega^2\bar{r}_{+1}}(1-\sqrt{1-\omega^2\bar{r}_{+1}^2})$
	$d\rho_{+1} = \frac{dr_{+1}}{\cos^2\frac{\omega r_{+1}}{2}} = \frac{2}{\omega^2\bar{r}_{+1}^2}\left(1-\frac{1}{\sqrt{1-\omega^2\bar{r}_{+1}^2}}\right)d\bar{r}_{+1}$

k=-1 FLRW radii [14, rel.(5.2),(5.8)]

r_{-1}	$r_{-1} = \frac{1}{\omega}\operatorname{arcsinh}(\omega\bar{r}_{-1}) = \frac{2}{\omega}\operatorname{arctanh}\frac{\omega\rho_{-1}}{2}$
	$dr_{-1} = \frac{d\bar{r}_{-1}}{\sqrt{1+\omega^2\bar{r}_{-1}^2}} = \frac{d\rho_{-1}}{1-\frac{\omega^2\rho_{-1}^2}{4}}$
$\bar{r}_{-1}$	$\bar{r}_{-1} = \frac{1}{\omega}\sinh(\omega r_{-1}) = \frac{\rho_{+1}}{1-\frac{\omega^2\rho_{-1}^2}{4}}$
	$d\bar{r}_{-1} = \cosh(\omega r_{-1})dr_{-1} = -4\frac{1+\frac{\omega^2\rho_{-1}^2}{4}}{\left(1-\frac{\omega^2\rho_{-1}^2}{4}\right)^2}d\rho_{-1}$
ρ_{-1}	$\rho_{-1} = \frac{2}{\omega}\tanh\frac{\omega r_{+1}}{2} = \frac{2}{\omega^2\bar{r}_{-1}}(\sqrt{1+\omega^2\bar{r}_{+1}^2}-1)$
	$d\rho_{-1} = \frac{dr_{-1}}{\cosh^2\frac{\omega r_{-1}}{2}} = \frac{2}{\omega^2\bar{r}_{-1}^2}\left(1-\frac{1}{\sqrt{1+\omega^2\bar{r}_{-1}^2}}\right)d\bar{r}_{-1}$

k=+1 FLRW time

t_{+1}	$t_{-1} = \operatorname{arctanh}(\cos(\omega\eta_{+1})) = \operatorname{arctanh}(\sin(\omega\eta'_{+1}))$
	$dt_{+1} = -\frac{d\eta_{+1}}{\sin(\omega\eta_{+1})} = \frac{d\eta'_{+1}}{\cos(\omega\eta'_{+1})}$

η_{+1}	$\eta_{+1} = \arccos(\tanh(\omega t_{+1})) = \frac{\pi}{2} - \eta'_{+1}$
	$d\eta_{+1} = -\frac{dt_{+1}}{\cosh(\omega t_{+1}} = -d\eta'_{+1}$
η'_{+1}	$\eta'_{+1} = \arcsin(\tanh(\omega t_{+1}) = \frac{\pi}{2} - \eta_{+1}$
	$d\eta'_{+1} = \frac{dt_{+1}}{\cosh(\omega t_{+1})} = -d\eta'_{+1}$

k=0 FLRW time

t	$t = \frac{1}{\omega}\ln(-\omega\eta)$
	$dt = -\frac{d\eta}{\omega\eta}$
η	$\eta = -\frac{1}{\omega}e^{-\omega t}$
	$d\eta = e^{-\omega t}dt$

k=-1 FLRW time

t_{-1}	$t_{-1} = -\operatorname{arcsinh}\frac{1}{\sinh(\omega\eta_{-1})}$
	$dt_{-1} = -\frac{d\eta_{-1}}{\sinh(\omega\eta_{-1})}$
η_{-1}	$\eta_{-1} = -\operatorname{arccoth}(\cosh(\omega t_{-1}))$
	$d\eta_{-1} = \frac{dt_{-1}}{\sinh(\omega t_{-1})}$

Note that there are many equivalent, but not so obvious ways to express t_{-1} and η_{-1}.

Table 3: **Proper comoving/ spatially flat FLRW chart**

	cartesian $\{t, \vec{x}\}$	spherical $\{t, r, \theta, \phi\}$
def.	$Z^0 = \frac{1}{\omega}\sinh(\omega t) + \frac{\omega}{2}e^{\omega t}\lvert\vec{x}\rvert^2$ $Z^i = e^{\omega t}x^i$ $Z^4 = \frac{1}{\omega}\cosh(\omega t) - \frac{\omega}{2}e^{\omega t}\lvert\vec{x}\rvert^2$	$Z^0 = \frac{1}{\omega}\sinh(\omega t) + \frac{\omega}{2}e^{\omega t}r^2$ $Z^1 = e^{\omega t}r\sin\theta\cos\phi$ $Z^2 = e^{\omega t}r\sin\theta\sin\phi$ $Z^3 = e^{\omega t}r\cos\theta$ $Z^4 = \frac{1}{\omega}\cosh(\omega t) - \frac{\omega}{2}e^{\omega t}r^2$
Ranges of	$t \in (-\infty, +\infty)$	$t \in (-\infty, +\infty)$
coordinates	$\vec{x} \in \mathbb{R}^3_x$	$r \in (0, +\infty)$
covers	region $Z^0 + Z^4 > 0$ (half of the manifold)	
Refs.	[17, p.130]	[4, rel. (2.1)]
	[20, rel. (13-14)]	[3, (A9)]
line elem.	$ds^2 = dt^2 - e^{2\omega t}d\vec{x}^2$	$ds^2 = dt^2 - e^{2\omega t}(dr^2 + r^2 d\Omega_2^2)$
metric	$\begin{cases} g_{00} = 1 \\ g_{0i} = g_{i0} = 0 \\ g_{ij} = e^{2\omega t}\eta_{ij} \end{cases}$ $\begin{cases} g^{00} = 1 \\ g^{0i} = g^{i0} = 0 \\ g^{ij} = e^{-2\omega t}\eta^{ij} \end{cases}$	
$\sqrt{\lvert g\rvert}$	$e^{3\omega t}$	
a tetrad	$\begin{cases} \hat{e}^{\hat{0}}_0 = 1 \\ \hat{e}^{\hat{0}}_i = \hat{e}^{\hat{i}}_0 = 0 \\ \hat{e}^{\hat{i}}_j = e^{\omega t}\delta^i_j \end{cases}$ $\begin{cases} e^0_{\hat{0}} = 1 \\ e^0_{\hat{i}} = e^i_{\hat{0}} = 0 \\ e^i_{\hat{j}} = e^{-\omega t}\delta^i_j \end{cases}$	
$\sqrt{\lvert g\rvert}$	$e^{3\omega t}$	

1st type Christoffel coeff. (9 non-null)	$\Gamma_{0ii} = \frac{\omega}{2}e^{\omega t}$ $\Gamma_{ii0} = \Gamma_{i0i} = -\frac{\omega}{2}e^{\omega t}$
2nd type Christoffel coeff. (9 non-null)	$\Gamma^0_{ii} = \omega e^{\omega t}$ $\Gamma^i_{i0} = \Gamma^i_{0i} = \frac{\omega}{2}$
Killing vectors	$K_{04} = \frac{1}{\omega}\partial_0 - x^i\partial_i$ $K_{0i} = -x_i(\partial_0 - \omega x^j\partial_j) + \frac{1}{2\omega}(1 + \omega^2\vec{x}^2 + e^{-\omega t})\partial_i$ $K_{i4} = x_i(\partial_0 - \omega x^j\partial_j) + \frac{1}{2\omega}(1 - \omega^2\vec{x}^2 - e^{-\omega t})\partial_i$ $K_{0i} + K_{i4} = \frac{1}{\omega}\partial_i$ $K_{ij} = x_i\partial_j - x_j\partial_i$

Table 4: **Conformal spatially flat FLRW chart**

	cartesian $\{\eta, \vec{x}\}$	spherical $\{\eta, r, \theta, \phi\}$
def.	$Z^0 = -\frac{1 - \omega^2(\eta^2 - \vec{x}^2)}{2\omega^2\eta}$ $Z^i = -\frac{x^i}{\omega\eta}$ $Z^4 = -\frac{1 + \omega^2(\eta^2 - \vec{x}^2)}{2\omega^2\eta}$	$Z^0 = -\frac{1 - \omega^2(\eta^2 - r^2)}{2\omega^2\eta}$ $Z^1 = -\frac{r\sin\theta\cos\phi}{\omega\eta}$ $Z^2 = -\frac{r\sin\theta\sin\phi}{\omega\eta}$ $Z^3 = -\frac{r\cos\theta}{\omega\eta}$ $Z^4 = -\frac{1 + \omega^2(\eta^2 - r^2)}{2\omega^2\eta}$
Ranges of coordinates	$\eta \in (-\infty, 0)$ $\vec{x} \in \mathbb{R}^3_x$	$\eta \in (-\infty, 0)$ $r \in (0, +\infty)$
cover	half of the manifold (same as spatially flat FLRW)	
Extended	$\eta \in (-\infty, +\infty)$	$\eta \in (-\infty, +\infty)$

coordinates	$\vec{x} \in \mathbb{R}^3_x$	$r \in (0, +\infty)$
cover	entire manifold, but not smoothly (with a singular surface)	
Refs.		[19, p.48] [45]
line elem.	$ds^2 = \frac{1}{\omega^2\eta^2}(d\eta^2 - d\vec{x}^2)$	$ds^2 = \frac{1}{\omega^2\eta^2}(d\eta^2 - dr^2 - r^2 d\Omega_2^2)$
metric	$g_{\mu\nu} = \frac{1}{\omega^2\eta^2}\eta_{\mu\nu}$ $g^{\mu\nu} = \omega^2\eta^2\eta^{\mu\nu}$	
$\sqrt{\lvert g \rvert}$	$\frac{1}{\omega^4\eta^4}$	
a tetrad	$\hat{e}^{\hat{\alpha}}_{\mu} = \frac{1}{\omega\eta}\delta^{\alpha}_{\mu}$ $e^{\mu}_{\hat{\alpha}} = \omega\eta\delta^{\mu}_{\alpha}$	
1st type Christoffel coeff. (10 non-null)	$\Gamma_{\mu 00} = -\dfrac{1}{\omega^2\eta^3}$ $\Gamma_{ii0} = \Gamma_{i0i} = \dfrac{1}{\omega^2\eta^3}$	
2nd type Christoffel coeff. (10 non-null)	$\Gamma^{\mu}_{00} = \Gamma^{i}_{i0} = \Gamma^{i}_{0i} = -\dfrac{1}{\eta}$	
Killing vectors	$K_{04} = -x^{\mu}\partial_{\mu}$ $K_{0i} = \omega x_i(x^{\nu}\partial_{\nu}) -$ $+ \dfrac{1}{2\omega}\left(1 - \omega^2(t^2 - \vec{x}^2)\right)\partial_i$ $K_{i4} = -\omega x_i(x^{\nu}\partial_{\nu}) +$ $+ \dfrac{1}{2\omega}\left(1 + \omega(t^2 - \vec{x}^2)\right)\partial_i$ $K_{0i} + K_{i4} = \dfrac{1}{\omega}\partial_i$ $K_{0i} - K_{i4} = 2\omega x_i(x^{\nu}\partial_{\nu}) -$ $- \omega(t^2 - \vec{x}^2)\partial_i$ $K_{ij} = x_i\partial_j - x_j\partial_i$	

Table 5: **k=+1 proper FLRW chart, standard coordinates**

	cartesian $\{t_{+1}, \vec{x}_{+1}\}$	spherical $\{t_{+1}, r_{+1}, \theta, \phi\}$
def.	$Z^0 = \dfrac{1}{\omega}\sinh(\omega t_{+1})$ $Z^i = \dfrac{1}{\omega\lvert\vec{x}_{+1}\rvert}\cosh(\omega t_{+1})\sin(\omega\lvert\vec{x}_{+1}\rvert)x^i_{+1}$ $Z^4 = \dfrac{1}{\omega}\cosh(\omega t_{+1})\cos(\omega\lvert\vec{x}_{+1}\rvert)$	$Z^0 = \dfrac{1}{\omega}\sinh(\omega t_{+1})$ $Z^1 = \dfrac{1}{\omega}\cosh(\omega t_{+1})\sin(\omega r_{+1})\sin\theta\cos\phi$ $Z^2 = \dfrac{1}{\omega}\cosh(\omega t_{+1})\sin(\omega r_{+1})\sin\theta\sin\phi$ $Z^3 = \dfrac{1}{\omega}\cosh(\omega t_{+1})\sin(\omega r_{+1})\cos\theta$ $Z^4 = \dfrac{1}{\omega}\cosh(\omega t_{+1})\cos(\omega r_{+1})$
Ranges of coordinates	$t_{+1} \in (-\infty, +\infty)$ $\lvert\vec{x}_{+1}\rvert \in \left[0, \frac{\pi}{\omega}\right)$	$t_{+1} \in (-\infty, +\infty)$ $r_{+1} \in \left[0, \frac{\pi}{\omega}\right)$
cover	all the manifold	
	comoving	
Refs.		[19, p.42] [18, p.124] [3, (16)] [20, (7,9)]
line elem.		$ds^2 = dt^2_{+1} - \cosh^2(\omega t_{+1})(dr^2_{+1} + \frac{\sin^2(\omega r_{+1})}{\omega^2}d\Omega^2_2)$
metric	$\begin{cases} g_{00} = 1 \\ g_{0i} = g_{i0} = 0 \\ g_{ij} = \dfrac{\cosh^2(\omega t_{+1})}{\omega^2\vec{x}^2_{+1}}\left[-x_{+1i}x_{+1j}\left(\omega^2 - \right.\right. \\ \qquad \left.\left. -\dfrac{\sin^2(\omega\lvert\vec{x}_{+1}\rvert)}{\vec{x}^2_{+1}}\right) + \eta_{ij}\sin^2(\omega\lvert\vec{x}_{+1}\rvert)\right] \end{cases}$ $\begin{cases} g^{00} = 1 \\ g^{0i} = g^{i0} = 0 \\ g^{ij} = \dfrac{\omega^2}{\cosh^2(\omega t_{+1})\sin^2(\omega\lvert\vec{x}_{+1}\rvert)}\left[x^i_{+1}x^j_{+1}\right. \\ \qquad \left.\left(\omega^2 - \dfrac{\sin^2(\omega\lvert\vec{x}_{+1}\rvert)}{\vec{x}^2_{+1}}\right) + \eta^{ij}\vec{x}^2_{+1}\right] \end{cases}$	

$\sqrt{\lvert g\rvert}$	$\frac{\cosh^3(\omega t_{+1})\sin^2(\omega\lvert\vec{x}_{+1}\rvert)}{\omega^3\vec{x}^2_{+1}}$
Killing vectors	$K_{ij} = x_i\partial_j - x_j\partial_i$

Table 6: **k=+1 proper FLRW chart, alternate coordinates**

	cartesian $\{t_{+1}, \vec{\bar{x}}_{+1}\}$	spherical $\{t_{+1}, \bar{r}_{+1}, \theta, \phi\}$
def.	$Z^0 = \frac{1}{\omega}\sinh(\omega t_{+1})$ $Z^i = \cosh(\omega t_{+1})\bar{x}^i_{+1}$ $Z^4 = \frac{1}{\omega}\cosh(\omega t_{+1})\sqrt{1-\omega^2\vec{\bar{x}}^2_{+1}}$	$Z^0 = \frac{1}{\omega}\sinh(\omega t_{+1})$ $Z^1 = \cosh(\omega t_{+1})\bar{r}_{+1}\sin\theta\cos\phi$ $Z^2 = \cosh(\omega t_{+1})\bar{r}_{+1}\sin\theta\sin\phi$ $Z^3 = \cosh(\omega t_{+1})\bar{r}_{+1}\cos\theta$ $Z^4 = \frac{1}{\omega}\cosh(\omega t_{+1})\sqrt{1-\omega^2\bar{r}^2_{+1}}$
Ranges of	$t_{+1} \in (-\infty, +\infty)$	$t_{+1} \in (-\infty, +\infty)$
coordinates	$\lvert\vec{\bar{x}}_{+1}\rvert \in \left[0, \frac{1}{\omega}\right)$	$\bar{r}_{+1} \in \left[0, \frac{1}{\omega}\right)$
cover	half of the manifold -corresponding to $r_{+1} \in \left[0, \frac{\pi}{2\omega}\right)$	
Refs.		[14, rel. (4.1)]
line elem.		$ds^2 = dt^2_{+1} - \cosh^2(\omega t_{+1})\left(\frac{d\bar{r}^2_{+1}}{1-\omega^2\bar{r}^2_{+1}} - \bar{r}^2_{+1}d\Omega^2_2\right)$
metric	$\begin{cases} g_{00} = 1 \\ g_{0i} = g_{i0} = 0 \\ g_{ij} = \cosh^2(\omega t_{+1})\left(\eta_{ij} + \dfrac{\omega^2\bar{x}^i_{+1}\bar{x}^j_{+1}}{1-\omega^2\vec{\bar{x}}^2_{+1}}\right) \end{cases}$ $\begin{cases} g^{00} = 1 \\ g^{0i} = g^{i0} = 0 \\ g^{ij} = \cosh^2(\omega t_{+1})(\eta^{ij} + \omega\bar{x}^i_{+1}\bar{x}^j_{+1}) \end{cases}$	
$\sqrt{\lvert g\rvert}$	$\frac{\cosh^3(\omega t_{+1})}{\sqrt{1-\omega^2\vec{\bar{x}}^2_{+1}}}$	

Killing vectors	$K_{04} = \sqrt{1-\omega^2\vec{x}^2}\left(\frac{1}{\omega}\partial_0 + \right.$ $\left. + \tanh(\omega t) x^i \partial_i\right)$ $K_{0i} = -x_i\partial_0 - \frac{\tanh(\omega t)}{\omega}\partial_i -$ $- \omega\tanh(\omega t) x_i x^j \partial_j$ $K_{i4} = \frac{1}{\omega}\sqrt{1-\omega^2\vec{x}^2}\partial_i$ $K_{ij} = x_i\partial_j - x_j\partial_i$

Table 7: **k=+1 proper FLRW chart, isotropic coordinates**

	cartesian $\{t_{+1}, \vec{\mathbf{x}}_{+1}\}$	spherical $\{t_{+1}, \rho_{+1}, \theta, \phi\}$
def.	$Z^0 = \frac{1}{\omega}\sinh(\omega t_{+1})$ $Z^i = \cosh(\omega t_{+1})\frac{\mathbf{x^i_{+1}}}{1+\frac{\omega^2\mathbf{x^2_{+1}}}{4}}$ $Z^4 = \frac{1}{\omega}\cosh(\omega t_{+1})\sqrt{\frac{1-\frac{\omega^2\mathbf{x^2_{+1}}}{4}}{1+\frac{\omega^2\mathbf{x^2_{+1}}}{4}}}$	$Z^0 = \frac{1}{\omega}\sinh(\omega t_{+1})$ $Z^1 = \cosh(\omega t_{+1})\frac{\rho_{+1}}{1+\frac{\omega^2\rho^2_{+1}}{4}}\sin\theta\cos\phi$ $Z^2 = \cosh(\omega t_{+1})\frac{\rho_{+1}}{1+\frac{\omega^2\rho^2_{+1}}{4}}\sin\theta\sin\phi$ $Z^3 = \cosh(\omega t_{+1})\frac{\rho_{+1}}{1+\frac{\omega^2\rho^2_{+1}}{4}}\cos\theta$ $Z^4 = \frac{1}{\omega}\cosh(\omega t_{+1})\sqrt{1-\frac{\omega^2\rho^2_{+1}}{1+\frac{\omega^2\rho^2_{+1}}{4}}}$
Ranges of coordinates	$t_{+1} \in (-\infty, +\infty)$ $\vec{\mathbf{x}}_{+1} \in \mathbb{R}^3$	$t_{+1} \in (-\infty, +\infty)$ $\rho_{+1} \in [0, \infty)$
cover	all the manifold speed of light is isotropic	
Refs.		[14, rel. (5.9)]
line elem.	$ds^2 = dt^2_{+1} - \frac{\cosh^2(\omega t_{+1})}{1+\frac{\omega^2\mathbf{x^2_{+1}}}{4}}d\vec{\mathbf{x}}^2$	$ds^2 = dt^2_{+1} - \frac{\cosh^2(\omega t_{+1})}{1+\frac{\omega^2\rho^2_{+1}}{4}}\left(d\rho^2_{+1} + \rho^2_{+1}d\Omega^2_2\right)$

metric	$\begin{cases} g_{00} = 1 \\ g_{0i} = g_{i0} = 0 \\ g_{ij} = \dfrac{\cosh^2(\omega t_{+1})}{1 + \frac{\omega^2 \mathbf{x}_{+1}^2}{4}} \eta_{ij} \end{cases}$ $\begin{cases} g^{00} = 1 \\ g^{0i} = g^{i0} = 0 \\ g^{ij} = \dfrac{1 + \frac{\omega^2 \mathbf{x}_{+1}^2}{4}}{\cosh^2(\omega t_{+1})} \eta^{ij} \end{cases}$
$\sqrt{\lvert g \rvert}$	$\frac{\cosh^3(\omega t_{+1})}{\left(1+\frac{\omega^2 \mathbf{x}_{+1}^2}{4}\right)^3}$
Killing vectors	$K_{i4} = \dfrac{1}{\omega}\left(1 - \dfrac{\omega^2 \vec{x}^2}{4}\right)\partial_i + \dfrac{\omega x_i}{2}(x^j \partial_j)$ $K_{ij} = x_i \partial_j - x_j \partial_i$

Table 8: **k=+1 conformal FLRW chart, standard coordinates**

	cartesian $\{\eta_{+1}, \vec{x}_{+1}\}$	spherical $\{\eta_{+1}, r_{+1}, \theta, \phi\}$
def.	$Z^0 = \dfrac{1}{\omega \tan(\omega \eta_{+1})}$ $Z^i = \dfrac{1}{\omega \lvert\vec{x}_{+1}\rvert \sin(\omega \eta_{+1})} \sin(\omega \lvert\vec{x}_{+1}\rvert) x^i_{+1}$ $Z^4 = \dfrac{1}{\omega \sin(\omega \eta_{+1})} \cos(\omega \lvert\vec{x}_{+1}\rvert)$	$Z^0 = \dfrac{1}{\omega \tan(\omega \eta_{+1})}$ $Z^1 = \dfrac{1}{\omega \sin(\omega \eta_{+1})} \sin(\omega r_{+1}) \sin\theta \cos\phi$ $Z^2 = \dfrac{1}{\omega \sin(\omega \eta_{+1})} \sin(\omega r_{+1}) \sin\theta \sin\phi$ $Z^3 = \dfrac{1}{\omega \sin(\omega \eta_{+1})} \sin(\omega r_{+1}) \cos\theta$ $Z^4 = \dfrac{1}{\omega \sin(\omega \eta_{+1})} \cos(\omega r_{+1})$
Ranges of coordinates	$\eta_{+1} \in \left[0, \frac{\pi}{\omega}\right)$ $\lvert\vec{x}_{+1}\rvert \in \left[0, \frac{\pi}{\omega}\right)$	$\eta_{+1} \in \left[0, \frac{\pi}{\omega}\right)$ $r_{+1} \in \left[0, \frac{\pi}{\omega}\right)$
cover	all the manifold	
	conformal comoving	
Refs.		[19, p.43] [17, p.134]

		[23][24]
line elem.		$ds^2=\frac{1}{\sin^2(\omega\eta_{+1})}(d\eta_{+1}^2-dr_{+1}^2-\frac{\sin^2(\omega r_{+1})}{\omega^2}d\Omega_2^2)$
metric	$\begin{cases} g_{00}=\dfrac{1}{\sin^2(\omega\eta_{+1})} \\ g_{0i}=g_{i0}=0 \\ g_{ij}=\dfrac{1}{\omega^2\sin^2(\omega\eta_{+1})\vec{x}_{+1}^2}\left[-x_{+1i}x_{+1j}\left(\omega^2- \right.\right. \\ \qquad \left.\left.-\dfrac{\sin^2(\omega\lvert\vec{x}_{+1}\rvert)}{\vec{x}_{+1}^2}\right)+\eta_{ij}\sin^2(\omega\lvert\vec{x}_{+1}\rvert)\right] \end{cases}$	
	$\begin{cases} g^{00}=\sin^2(\omega\eta_{+1}) \\ g^{0i}=g^{i0}=0 \\ g^{ij}=\dfrac{\omega^2\sin^2(\omega\eta_{+1})}{\sin^2(\omega\lvert\vec{x}_{+1}\rvert)}\left[x_{+1}^i x_{+1}^j\right. \\ \qquad \left.\left(\omega^2-\dfrac{\sin^2(\omega\lvert\vec{x}_{+1}\rvert)}{\vec{x}_{+1}^2}\right)+\eta^{ij}\vec{x}_{+1}^2\right] \end{cases}$	
$\sqrt{\lvert g\rvert}$	$\frac{\sin^2(\omega\lvert\vec{x}_{+1}\rvert)}{\omega^3\sin^4(\omega t_{+1})\vec{x}_{+1}^2}$	
Killing vectors	$K_{ij}=x_i\partial_j-x_j\partial_i$	

Table 9: **k=+1 conformal FLRW chart, standard alternate coordinates**

	cartesian $\{\eta'_{+1},\vec{x}_{+1}\}$	spherical $\{\eta'_{+1},r_{+1},\theta,\phi\}$
def.	$Z^0=\frac{1}{\omega}\tan(\omega\eta'_{+1})$ $Z^i=\frac{1}{\omega\lvert\vec{x}_{+1}\rvert\cos(\omega\eta'_{+1})}\sin(\omega\lvert\vec{x}_{+1}\rvert)x_{+1}^i$ $Z^4=\frac{1}{\omega\cos(\omega\eta'_{+1})}\cos(\omega\lvert\vec{x}_{+1}\rvert)$	$Z^0=\frac{1}{\omega}\tan(\omega\eta'_{+1})$ $Z^1=\frac{1}{\omega\cos(\omega\eta'_{+1})}\sin(\omega r_{+1})\sin\theta\cos\phi$ $Z^2=\frac{1}{\omega\cos(\omega\eta'_{+1})}\sin(\omega r_{+1})\sin\theta\sin\phi$ $Z^3=\frac{1}{\omega\cos(\omega\eta'_{+1})}\sin(\omega r_{+1})\cos\theta$ $Z^4=\frac{1}{\omega\cos(\omega\eta'_{+1})}\cos(\omega r_{+1})$
Ranges of	$\eta'_{+1}\in\left[-\frac{\pi}{2\omega},\frac{\pi}{2\omega}\right)$	$\eta'_{+1}\in\left[-\frac{\pi}{2\omega},\frac{\pi}{2\omega}\right)$

coordinates	$\lvert\vec{x}_{+1}\rvert \in \left[0, \frac{\pi}{\omega}\right)$	$r_{+1} \in \left[0, \frac{\pi}{\omega}\right)$
cover	all the manifold	
	conformal comoving	
Refs.		[14, rel. (5.16)]
		[18, p.126, cos instead of cosh in metric]
line elem.		$ds^2 = \frac{1}{\cos^2(\omega\eta'_{+1})}(d\eta'^2_{+1} - dr^2_{+1} - \frac{\sin^2(\omega r_{+1})}{\omega^2} d\Omega_2^2)$
metric	$\begin{cases} g_{00} = \dfrac{1}{\cos^2(\omega\eta'_{+1})} \\ g_{0i} = g_{i0} = 0 \\ g_{ij} = \dfrac{1}{\omega^2\cos^2(\omega\eta'_{+1})\vec{x}^2_{+1}} \left[-x_{+1i}x_{+1j}\left(\omega^2 - \right.\right. \\ \qquad \left.\left. - \dfrac{\sin^2(\omega\lvert\vec{x}_{+1}\rvert)}{\vec{x}^2_{+1}}\right) + \eta_{ij}\sin^2(\omega\lvert\vec{x}_{+1}\rvert)\right] \end{cases}$	
	$\begin{cases} g^{00} = \cos^2(\omega\eta'_{+1}) \\ g^{0i} = g^{i0} = 0 \\ g^{ij} = \dfrac{\omega^2\cos^2(\omega\eta'_{+1})}{\sin^2(\omega\lvert\vec{x}_{+1}\rvert)} \left[x^i_{+1}x^j_{+1}\right. \\ \qquad \left.\left(\omega^2 - \dfrac{\sin^2(\omega\lvert\vec{x}_{+1}\rvert)}{\vec{x}^2_{+1}}\right) + \eta^{ij}\vec{x}^2_{+1}\right] \end{cases}$	
$\sqrt{\lvert g\rvert}$	$\frac{\sin^2(\omega\lvert\vec{x}_{+1}\rvert)}{\omega^3\sin^4(\omega t_{+1})\vec{x}^2_{+1}}$	
Killing vectors	$K_{ij} = x_i\partial_j - x_j\partial_i$	

Table 10: **k=+1 conformal FLRW chart, alternate standard coordinates**

	cartesian $\{\eta'_{+1}, \vec{x}'_{+1}\}$	spherical $\{\eta'_{+1}, r'_{+1}, \theta, \phi\}$

def.	$Z^0 = \frac{1}{\omega}\tan(\omega\eta'_{+1})$ $Z^i = \frac{1}{\omega\lvert\vec{x}_{+1}\rvert\cos(\omega\eta'_{+1})}\cos(\omega\lvert\vec{x}'_{+1}\rvert)x'^i_{+1}$ $Z^4 = \frac{1}{\omega\cos(\omega\eta'_{+1})}\sin(\omega\lvert\vec{x}'_{+1}\rvert)$	$Z^0 = \frac{1}{\omega}\tan(\omega\eta'_{+1})$ $Z^1 = \frac{1}{\omega\cos(\omega\eta'_{+1})}\cos(\omega r'_{+1})\sin\theta\cos\phi$ $Z^2 = \frac{1}{\omega\cos(\omega\eta'_{+1})}\cos(\omega r'_{+1})\sin\theta\sin\phi$ $Z^3 = \frac{1}{\omega\cos(\omega\eta'_{+1})}\cos(\omega r'_{+1})\cos\theta$ $Z^4 = \frac{1}{\omega\cos(\omega\eta'_{+1})}\sin(\omega r'_{+1})$
Ranges of coordinates	$\eta'_{+1} \in \left[-\frac{\pi}{2\omega}, \frac{\pi}{2\omega}\right)$ $\lvert\vec{x}'_{+1}\rvert \in \left[-\frac{\pi}{2\omega}, \frac{\pi}{2\omega}\right)$	$\eta'_{+1} \in \left[-\frac{\pi}{2\omega}, \frac{\pi}{2\omega}\right)$ $r_{+1} \in \left[-\frac{\pi}{2\omega}, \frac{\pi}{2\omega}\right)$
cover	all the manifold	
	conformal comoving	
Refs.		[14, rel. (3.18)]
line elem.		$ds^2 = \frac{1}{\cos^2(\omega\eta'_{+1})}(d\eta'^2_{+1} - dr'^2_{+1} - \frac{\cos^2(\omega r'_{+1})}{\omega^2}d\Omega_2^2)$

Table 11: **k=-1 proper FLRW chart, standard coordinates**

	cartesian $\{t_{-1}, \vec{x}_{-1}\}$	spherical $\{t_{-1}, r_{-1}, \theta, \phi\}$
def.	$Z^0 = \frac{1}{\omega}\sinh(\omega t_{-1})\cosh(\omega\lvert\vec{x}_{-1}\rvert)$ $Z^i = \frac{1}{\omega\lvert\vec{x}_{+1}\rvert}\sinh(\omega t_{-1})\sinh(\omega\lvert\vec{x}_{+1}\rvert)x^i_{-1}$ $Z^4 = \frac{1}{\omega}\cosh(\omega t_{-1})$	$Z^0 = \frac{1}{\omega}\sinh(\omega t_{-1})\cosh(\omega r_{-1})$ $Z^1 = \frac{1}{\omega}\sinh(\omega t_{-1})\sinh(\omega r_{-1})\sin\theta\cos\phi$ $Z^2 = \frac{1}{\omega}\sinh(\omega t_{-1})\sinh(\omega r_{-1})\sin\theta\sin\phi$ $Z^3 = \frac{1}{\omega}\sinh(\omega t_{-1})\sinh(\omega r_{-1})\cos\theta$ $Z^4 = \frac{1}{\omega}\cosh(\omega t_{-1})$
Ranges of coordinates	$t_{-1} \in (-\infty, +\infty)$ $\lvert\vec{x}_{-1}\rvert \in [0, \infty)$	$t_{-1} \in (-\infty, +\infty)$ $r_{-1} \in [0, \infty)$
	comoving	
Refs.		[19, p.50] [3, (A6,A7)] [20, (25,26)]
line elem.		$ds^2 = dt^2_{-1} - \sinh^2(\omega t_{-1})(dr^2_{-1} + \frac{\sinh^2(\omega r_{-1})}{\omega^2}d\Omega^2_2)$
$\sqrt{\lvert g\rvert}$	$\frac{\sinh^3(\omega t_{-1})\sinh^2(\omega\lvert\vec{x}_{-1}\rvert)}{\omega^3\vec{x}^2_{-1}}$	

Table 12: **k=-1 proper FLRW chart, alternate coordinates**

	cartesian $\{t_{-1}, \bar{\vec{x}}_{-1}\}$	spherical $\{t_{-1}, \bar{r}_{-1}, \theta, \phi\}$
def.	$Z^0 = \frac{\sqrt{1+\omega^2\bar{r}^2_{-1}}}{\omega}\sinh(\omega t_{-1})$ $Z^i = \sinh(\omega t_{-1})\bar{x}^i_{-1}$ $Z^4 = \frac{1}{\omega}\cosh(\omega t_{-1})$	$Z^0 = \frac{\sqrt{1+\omega^2\bar{r}^2_{-1}}}{\omega}\sinh(\omega t_{-1})$ $Z^1 = \sinh(\omega t_{-1})\bar{r}_{-1}\sin\theta\cos\phi$ $Z^2 = \sinh(\omega t_{-1})\bar{r}_{-1}\sin\theta\sin\phi$ $Z^3 = \sinh(\omega t_{-1})\bar{r}_{-1}\cos\theta$ $Z^4 = \frac{1}{\omega}\cosh(\omega t_{-1})$

Ranges of coordinates	$t_{-1} \in (-\infty, +\infty)$ $\lvert\vec{\bar{x}}_{-1}\rvert \in [0, \infty)$	$t_{-1} \in (-\infty, +\infty)$ $\bar{r}_{-1} \in [0, \infty)$
Refs.		[14, rel. (4.3)]
line elem.		$ds^2 = dt_{-1}^2 - \sinh^2(\omega t_{-1})\left(\frac{d\bar{r}_{-1}^2}{1+\omega^2\bar{r}_{-1}^2} - \bar{r}_{-1}^2 d\Omega_2^2\right)$
$\sqrt{\lvert g\rvert}$	$\frac{\sinh^3(\omega t_{-1})}{\sqrt{1+\omega^2\vec{\bar{x}}_{-1}^2}}$	
Killing vectors	$K_{04} = \sqrt{1+\omega^2\vec{x}^2}\left(\frac{1}{\omega}\partial_0 + + \coth(\omega t)x^i\partial_i\right)$ $K_{0i} = \frac{1}{\omega}\sqrt{1+\omega^2\vec{x}^2}\partial_i$ $K_{i4} = x_i\partial_0 - \frac{\coth(\omega t)}{\omega}\partial_i + + \omega\coth(\omega t)x_i x^j\partial_j$ $K_{ij} = x_i\partial_j - x_j\partial_i$	

Table 13: **k=-1 proper FLRW chart, isotropic coordinates**

	cartesian $\{t_{-1}, \vec{\mathbf{x}}_{-1}\}$	spherical $\{t_{-1}, \rho_{-1}, \theta, \phi\}$
def.	$Z^0 = \frac{1}{\omega}\cosh(\omega t_{-1})\sqrt{\frac{1+\frac{\omega^2\mathbf{x_{-1}^2}}{4}}{1-\frac{\omega^2\mathbf{x_{-1}^2}}{4}}}$ $Z^i = \sinh(\omega t_{-1})\frac{\mathbf{x_{-1}^i}}{1-\frac{\omega^2\mathbf{x_{-1}^2}}{4}}$ $Z^4 = \frac{1}{\omega}\cosh(\omega t_{-1})$	$Z^0 = \frac{1}{\omega}\cosh(\omega t_{-1})\sqrt{1+\frac{\omega^2\rho_{-1}^2}{1-\frac{\omega^2\rho_{-1}^2}{4}}}$ $Z^1 = \sinh(\omega t_{-1})\frac{\rho_{-1}}{1-\frac{\omega^2\rho_{-1}^2}{4}}\sin\theta\cos\phi$ $Z^2 = \sinh(\omega t_{-1})\frac{\rho_{-1}}{1-\frac{\omega^2\rho_{-1}^2}{4}}\sin\theta\sin\phi$ $Z^3 = \sinh(\omega t_{-1})\frac{\rho_{-1}}{1-\frac{\omega^2\rho_{-1}^2}{4}}\cos\theta$ $Z^4 = \frac{1}{\omega}\cosh(\omega t_{-1})$
Ranges of coordinates	$t_{-1} \in (-\infty, +\infty)$ $\vec{\mathbf{x}}_{-1} \in \mathbb{R}^3$	$t_{-1} \in (-\infty, +\infty)$ $\rho_{-1} \in [0, \infty)$

speed of light is isotropic

Refs.		[14, rel. (5.9)]
line elem.	$ds^2 = dt_{-1}^2 - \frac{\sinh^2(\omega t_{-1})}{1-\frac{\omega^2 \mathbf{x_{-1}^2}}{4}} d\vec{\mathbf{x}}^2$	$ds^2 = dt_{-1}^2 - \frac{\sinh^2(\omega t_{-1})}{1-\frac{\omega^2 \rho_{-1}^2}{4}} \left(d\rho_{-1}^2 + \rho_{-1}^2 d\Omega_2^2\right)$
metric	$\begin{cases} g_{00} = 1 \\ g_{0i} = g_{i0} = 0 \\ g_{ij} = \dfrac{\sinh^2(\omega t_{-1})}{1-\frac{\omega^2 \mathbf{x_{-1}^2}}{4}} \eta_{ij} \end{cases}$ $\begin{cases} g^{00} = 1 \\ g^{0i} = g^{i0} = 0 \\ g^{ij} = \dfrac{1-\frac{\omega^2 \mathbf{x_{-1}^2}}{4}}{\sinh^2(\omega t_{-1})} \eta^{ij} \end{cases}$	
$\sqrt{\lvert g \rvert}$	$\frac{\sinh^3(\omega t_{-1})}{\left(1-\frac{\omega^2 \mathbf{x_{-1}^2}}{4}\right)^3}$	

Table 14: **k=-1 conformal FLRW chart, standard coordinates**

	cartesian $\{\eta_{-1}, \vec{x}_{-1}\}$	spherical $\{\eta_{-1}, r_{-1}, \theta, \phi\}$
def.	$Z^0 = -\dfrac{1}{\omega \sinh(\omega \eta_{-1})} \cosh(\omega \lvert \vec{x}_{-1} \rvert)$ $Z^i = -\dfrac{1}{\omega \lvert \vec{x}_{+1} \rvert \sinh(\omega \eta_{-1})} \sinh(\omega \lvert \vec{x}_{+1} \rvert) x_{-1}^i$ $Z^4 = -\dfrac{1}{\omega \tanh(\omega t_{-1})}$	$Z^0 = -\dfrac{1}{\omega \cosh(\omega \eta_{-1})} \cosh(\omega r_{-1})$ $Z^1 = -\dfrac{1}{\omega \cosh(\omega \eta_{-1})} \sinh(\omega r_{-1}) \sin\theta \cos\phi$ $Z^2 = -\dfrac{1}{\omega \cosh(\omega \eta_{-1})} \sinh(\omega r_{-1}) \sin\theta \sin\phi$ $Z^3 = -\dfrac{1}{\omega \cosh(\omega \eta_{-1})} \sinh(\omega r_{-1}) \cos\theta$ $Z^4 = -\dfrac{1}{\omega \tanh(\omega \eta_{-1})}$
Ranges of	$\eta_{-1} \in (-\infty, 0)$	$\eta_{-1} \in (-\infty, 0)$
coordinates	$\lvert \vec{x}_{-1} \rvert \in [0, \infty)$	$r_{-1} \in [0, \infty)$

comoving

Refs.		[16, rel. (4.9)] [14, rel. (5.18)]
line elem.		$ds^2 = \frac{1}{\sinh^2(\omega\eta_{-1})}(d\eta^2_{-1} - dr^2_{-1} - $ $- \frac{\sinh^2(\omega r_{-1})}{\omega^2} d\Omega^2_2)$
$\sqrt{\lvert g\rvert}$	$\frac{\sinh^2(\omega\lvert\vec{x}_{-1}\rvert)}{\omega^3 \sinh^3(\omega\eta_{-1})\vec{x}^2_{-1}}$	

Table 15: **Overview- standard proper FLRW charts**

$\times$	line element: $ds^2 = dt_k^2 - a^2(t_k)(dr_k^2 + f^2(r_k)d\Omega_2^2)$		
spatial curvature	$k = +1$	$k = 0$	$k = -1$
name	closed	spatially flat	open
introduced:	Lanczos, 1922 [40]	Lemâitre, 1925 [41]	Robertson, 1933 [39]
studied:		Robertson, 1928 [38]	
slices of const t_k	spheres	planes	hyperboloids
scale factor $a(t_k)$	$\cosh(\omega t_{+1})$	$e^{\omega t}$	$\sinh(\omega t_{-1})$
$f(r_k)$	$\frac{\sin(\omega r_{+1})}{\omega}$	r	$\frac{\sinh(\omega r_{-1})}{\omega}$
	$Z^0 = \frac{1}{\omega}\sinh(\omega t_{+1})$	$Z^0 = \frac{1}{\omega}\sinh(\omega t) + \frac{\omega}{2}e^{\omega t}r^2$	$Z^0 = \frac{1}{\omega}\sinh(\omega t_{-1})\cosh(\omega r_{-1})$
	$Z^1 = \frac{1}{\omega}\cosh(\omega t_{+1})\sin(\omega r_{+1})\sin\theta\cos\phi$	$Z^1 = e^{\omega t}r\sin\theta\cos\phi$	$Z^3 = \frac{1}{\omega}\sinh(\omega t_{-1})\sinh(\omega r_{-1})\sin\theta\sin\phi$
	$Z^2 = \frac{1}{\omega}\cosh(\omega t_{+1})\sin(\omega r_{+1})\sin\theta\sin\phi$	$Z^2 = e^{\omega t}r\sin\theta\sin\phi$	$Z^2 = \frac{1}{\omega}\sinh(\omega t_{-1})\sinh(\omega r_{-1})\sin\theta\cos\phi$
	$Z^3 = \frac{1}{\omega}\cosh(\omega t_{+1})\sin(\omega r_{+1})\cos\theta$	$Z^3 = e^{\omega t}r\cos\theta$	$Z^1 = \frac{1}{\omega}\sinh(\omega t_{-1})\sinh(\omega r_{-1})\cos\theta$
	$Z^4 = \frac{1}{\omega}\cosh(\omega t_{+1})\cos(\omega r_{+1})$	$Z^4 = \frac{1}{\omega}\cosh(\omega t) - \frac{\omega}{2}e^{\omega t}r^2$	$Z^4 = \frac{1}{\omega}\cosh(\omega t_{-1})$
line element	$ds^2 = dt_{+1}^2 - \cosh^2(\omega t_{+1})$	$ds^2 = dt^2 - e^{2\omega t}$	$ds^2 = dt_{+1}^2 - \sinh^2(\omega t_{-1})$
	$(dr_{+1}^2 + \frac{\sin^2(\omega r_{+1})}{\omega^2}d\Omega_2^2)$	$(dr^2 + r^2 d\Omega_2^2)$	$(dr_{-1}^2 + \frac{\sinh^2(\omega r_{-1})}{\omega^2}d\Omega_2^2)$
time range	$t_{+1} \in (-\infty, +\infty)$	$t \in (-\infty, +\infty)$	$t_{-1} \in (-\infty, +\infty)$
radius range	$r_{+1} \in \left[0, \frac{\pi}{\omega}\right)$	$r \in [0, +\infty)$	$r_{-1} \in [0, +\infty)$
cover	all dS	half of dS ($Z^0 + Z^4 > 0$)	a quarter

Table 16: **Overview- standard conformal FLRW charts**

$\times$	line element: $ds^2 = \Omega^2(\eta_k)(d\eta_k^2 - dr_k^2 - f^2(r_k)d\Omega_2^2)$ [18, p.139]		
conformal time	$\tan\frac{\omega\eta_{+1}}{2} = e^{\omega t_{+1}}$	$\eta = -\frac{1}{\omega}e^{-\omega t}$	$\coth\frac{\omega\eta_{-1}}{2} = -e^{\omega t_{-1}}$
$\eta_k(t_k) = \int\frac{dt_k}{a(t_k)}$	$\sin(\omega\eta_{+1}) = \text{sech}(\omega t_{+1})$ †		$e^{\omega\eta_{-1}} = \tanh\frac{\omega t_{-1}}{2}$

conformal	$ds^2 = \frac{1}{\sin^2\eta_{+1}}(d\eta_{+1}^2 -$	$ds^2 = \frac{1}{\omega^2\eta^2}(d\eta^2 -$	$ds^2 = \frac{1}{\sinh^2\eta_{-1}}(d\eta_{-1}^2 -$
line element	$-dr_{+1}^2 - \frac{\sin^2 r_{+1}}{\omega^2} d\Omega_2^2)$	$-dr^2 - r^2 d\Omega_2^2)$	$-dr_{-1}^2 - \frac{\sinh^2 r_{-1}}{\omega^2} d\Omega_2^2)$
chart is	part of Einstein	part of Minkowski	part of Open
conformal to	Static Universe	spacetime	Einstein Universe
with conf. factor	$\frac{1}{\omega \sin\eta_{+1}}$	$\frac{1}{\omega\eta_0}$	$\frac{1}{\omega \sinh\eta_{-1}}$
range	$\eta_{+1} \in \left(0, \frac{\pi}{\omega}\right)$	$\eta \in (-\infty, 0)$	$\eta_{-1} \in (-\infty, 0)$
cover	entire manifold	half of the manifold	
extended range	$-$	$\eta \in (-\infty, +\infty)$	
cover (not smoothly)	$-$	entire manifold	

3 Static charts

Table 17: **Static radial coordinates on de Sitter spacetime**

r_s	$r_s = \frac{\bar{r}_s}{\sqrt{1+\omega^2\bar{r}_s^2}} = \frac{1}{\omega}\sin(\omega r_F) = \frac{1}{\omega}\tanh(\omega r^*) = \frac{\rho_s}{1+\frac{\omega^2\rho_s^2}{4}}$
	$dr_s = \frac{d\bar{r}_s}{(\sqrt{1+\omega^2\bar{r}_s^2})^{\frac{3}{2}}} = \cos(\omega r_F)dr_F = \frac{dr^*}{\cosh(\omega r^*)} = 4\frac{1-\frac{\omega^2\rho_s^2}{4}}{1+\frac{\omega^2\rho_s^2}{4}}d\rho_s$
$\bar{r}_s$	$\bar{r}_s = \frac{r_s}{\sqrt{1-\omega^2 r_s^2}} = \frac{1}{\omega}\tan(\omega r_F) = \frac{1}{\omega}\sinh(\omega r^*) = \frac{\rho_s}{1-\frac{\omega^2\rho_s^2}{4}}$
	$d\bar{r}_s = \frac{dr_s}{(\sqrt{1-\omega^2 r_s^2})^{\frac{3}{2}}} = \frac{dr_F}{\cos(\omega r_F)} = \cosh(\omega r^*)dr^* = -4\frac{1+\frac{\omega^2\rho_s^2}{4}}{1-\frac{\omega^2\rho_s^2}{4}}d\rho_s$
r_F	$r_F = \frac{1}{\omega}\arcsin(\omega r_s) = \frac{1}{\omega}\arcsin\frac{\omega\bar{r}_s}{\sqrt{1+\omega^2\bar{r}_s^2}} = \frac{1}{\omega}\arcsin(\tanh(\omega r^*)) = \frac{1}{\omega}\arcsin\frac{\omega\rho_s}{1+\frac{\omega^2\rho_s^2}{4}}$
	$dr_F = \frac{dr_s}{\sqrt{1-\omega^2 r_s^2}} = \frac{d\bar{r}_s}{1+\omega^2\bar{r}_s^2} = \frac{dr^*}{\cosh(\omega r^*)} = -\frac{d\rho_s}{1+\frac{\omega^2\rho_s^2}{4}}$
r^*	$r^* = \frac{1}{\omega}\operatorname{arctanh}(\omega r_s) = \frac{1}{\omega}\operatorname{arctanh}\frac{\omega\bar{r}_s}{\sqrt{1+\omega^2\bar{r}_s^2}} = \frac{1}{\omega}\operatorname{arctanh}(\sin(\omega r^*)) = \frac{1}{\omega}\operatorname{arctanh}\frac{\omega\rho_s}{1+\frac{\omega^2\rho_s^2}{4}}$
	$dr^* = \frac{dr_s}{1-\omega^2 r_s^2} = \frac{d\bar{r}_s}{\sqrt{1+\omega^2\bar{r}_s^2}} = \frac{dr_F}{\cos(\omega r_F)} = \frac{d\rho_s}{1-\frac{\omega^2\rho_s^2}{4}}$
ρ_s	$\rho_s = \frac{2}{\omega^2 r_s}(1-\sqrt{1-\omega^2 r_s^2}) = \frac{2}{\omega^2\bar{r}_s}(\sqrt{1+\omega^2\bar{r}_s^2}-1) = \frac{2}{\omega}\tan\frac{\omega r_F}{2} = \frac{2}{\omega}\tanh\frac{\omega r^*}{2}$
	$d\rho_s = \frac{2dr_s}{\omega^2 r_s^2}\left(\frac{1}{\sqrt{1-\omega^2 r_s^2}}-1\right) = \frac{2d\bar{r}_s}{\omega^2\bar{r}_s^2}\left(\sqrt{1-\frac{1}{\omega^2\bar{r}_s^2}}\right) = \frac{dr_F}{\cos^2(\omega r_F)} = \frac{dr^*}{\cosh^2(\omega r^*)}$

Table 18: **Static chart**

Introduced:	Eddington, 1922 [37]	
	cartesian $\{t_s, \vec{x}_s\}$	spherical $\{t_s, r_s, \theta, \phi\}$
def.	$Z^0 = \frac{\sqrt{1-\omega^2\vec{x}_s^2}}{\omega}\sinh(\omega t_s)$ $Z^i = x_s^i$ $Z^4 = \frac{\sqrt{1-\omega^2\vec{x}_s^2}}{\omega}\cosh(\omega t_s)$	$Z^0 = \frac{\sqrt{1-\omega^2 r_s^2}}{\omega}\sinh(\omega t_s)$ $Z^1 = r_s\sin\theta\cos\phi$ $Z^2 = r_s\sin\theta\sin\phi$ $Z^3 = r_s\cos\theta$ $Z^4 = \frac{\sqrt{1-\omega^2 r_s^2}}{\omega}\cosh(\omega t_s)$

Ranges of coordinates	$t_s \in (-\infty, +\infty)$ $\lvert\vec{x}_s\rvert \in \left[0, \frac{1}{\omega}\right)$	$t_s \in (-\infty, +\infty)$ $r_s \in \left[0, \frac{1}{\omega}\right)$
cover	region $Z^0 + Z^1 > 0$ (half of the manifold) has coordinate singularity at $\lvert\vec{x}_s\rvert = r_s = \frac{1}{\omega}$ (event horizon for the observer at the origin) generator of t is a Killing vector	
Refs.		[4, rel. (2.5)] [20, (15-16)]
line elem.	$ds^2 = (1-\omega^2\vec{x}_s^2)dt_s^2 - \frac{\omega^2(\vec{x}_s d\vec{x}_s)^2}{1-\omega^2\vec{x}_s^2} - d\vec{x}_s^{\,2}$	$ds^2 = (1-\omega^2 r_s^2)dt_s^2 - \frac{dr_s^2}{1-\omega^2 r_s^2} - r_s^2 d\Omega_2^2$
metric	$\begin{cases} g_{00} = 1-\omega^2\vec{x}_s^2 \\ g_{0i} = g_{i0} = 0 \\ g_{ij} = \eta_{ij} - \dfrac{\omega^2 x_{si}x_{sj}}{1-\omega^2\vec{x}_s^2} \end{cases}$ $\begin{cases} g^{00} = \dfrac{1}{1-\omega^2\vec{x}_s^2} \\ g^{0i} = g^{i0} = 0 \\ g^{ij} = \eta^{ij} + \omega x_s^i x_s^j \end{cases}$	
$\sqrt{\lvert g\rvert}$	1	
a tetrad	$\begin{cases} \hat{e}_0^{\hat{0}} = \sqrt{1-\omega^2\vec{x}_s^2} \\ \hat{e}_i^{\hat{0}} = \hat{e}_0^{\hat{i}} = 0 \\ \hat{e}_j^{\hat{i}} = \delta_j^i - \dfrac{x_s^i x_{sj}}{\vec{x}^2}\left(\sqrt{1-\omega^2\vec{x}_s^2} - 1\right) \end{cases}$ $\begin{cases} e_{\hat{0}}^0 = \dfrac{1}{\sqrt{1-\omega^2\vec{x}_s^2}} \\ e_{\hat{i}}^0 = e_{\hat{0}}^i = 0 \\ e_{\hat{j}}^i = \delta_j^i - \dfrac{x_s^i x_{sj}}{\vec{x}^2}\left(\dfrac{1}{\sqrt{1-\omega^2\vec{x}_s^2}} - 1\right) \end{cases}$	

1st type Christoffel coeff. (36 non-null)	$$\Gamma_{00i} = \Gamma_{0i0} = -\omega^2 x_s^i$$ $$\Gamma_{0ii} = \omega^2 x_s^i$$ $$\Gamma_{ijk} = -\frac{\omega^2 x_s^i x_s^j x_s^k}{(1-\omega^2 \vec{x}_s^2)^2} - \frac{\omega x_s^i}{1-\omega^2 \vec{x}_s^2}\delta_k^j$$
2nd type Christoffel coeff. (36 non-null)	$$\Gamma^0_{0i} = \Gamma^0_{i0} = -\frac{\omega^2 x_s^i}{1-\omega^2 \vec{x}_s^2}$$ $$\Gamma^0_{ii} = -\omega^2 x_s^i (1-\omega^2 \vec{x}_s^2)$$ $$\Gamma^i_{jk} = \frac{\omega^2 x_s^i x_s^j x_s^k}{1-\omega^2 \vec{x}_s^2} + \omega x_s^i \delta_k^j$$
Killing vectors	$$K_{04} = \frac{1}{\omega}\partial_0$$ $$K_{0i} = -\frac{x_i \cosh(\omega t)}{\sqrt{1-\omega^2 \vec{x}_s^2}}\partial_0 + \frac{\sinh(\omega t)\sqrt{1-\omega^2 \vec{x}_s^2}}{\omega}\partial_i$$ $$K_{i4} = -\frac{x_i \sinh(\omega t)}{\sqrt{1-\omega^2 \vec{x}_s^2}}\partial_0 - \frac{\cosh(\omega t)\sqrt{1-\omega^2 \vec{x}_s^2}}{\omega}\partial_i$$ $$K_{0i} + K_{i4} = -\frac{x_i e^{\omega t}}{\sqrt{1-\omega^2 \vec{x}_s^2}}\partial_0 - \frac{e^{\omega t}\sqrt{1-\omega^2 \vec{x}_s^2}}{\omega}\partial_i$$ $$K_{0i} - K_{i4} = -\frac{x_i e^{-\omega t}}{\sqrt{1-\omega^2 \vec{x}_s^2}}\partial_0 + \frac{e^{\omega t}\sqrt{1-\omega^2 \vec{x}_s^2}}{\omega}\partial_i$$ $$K_{ij} = x_i\partial_j - x_j\partial_i$$

Table 19: **Alternate de Sitter static chart**

Introduced:	de Sitter, 1917 [42] [43]	
	cartesian $\{t_s, \vec{\bar{x}}_s\}$	spherical $\{t_s, \bar{r}_s, \theta, \phi\}$

def.	$Z^0 = \frac{1}{\omega\sqrt{1+\omega^2\bar{\vec{x}}_s^2}}\cosh(\omega t_s)$ $Z^i = \frac{\bar{x}_s^i}{\sqrt{1+\omega^2\bar{\vec{x}}_s^2}}$ $Z^4 = \frac{1}{\omega\sqrt{1+\omega^2\bar{\vec{x}}_s^2}}\sinh(\omega t_s)$	$Z^0 = \frac{1}{\omega\sqrt{1+\omega^2\bar{\vec{r}}_s^2}}\cosh(\omega t_s)$ $Z^1 = \frac{\bar{r}_s}{\sqrt{1+\omega^2\bar{\vec{r}}_s^2}}\sin\theta\cos\phi$ $Z^2 = \frac{\bar{r}_s}{\sqrt{1+\omega^2\bar{\vec{r}}_s^2}}\sin\theta\sin\phi$ $Z^3 = \frac{\bar{r}_s}{\sqrt{1+\omega^2\bar{\vec{r}}_s^2}}\cos\theta$ $Z^4 = \frac{1}{\omega\sqrt{1+\omega^2\bar{\vec{r}}_s^2}}\sinh(\omega t_s)$
Ranges of coordinates	$t_s \in (-\infty, +\infty)$ $\lvert\bar{\vec{x}}_s\rvert \in \left[0, \frac{1}{\omega}\right)$	$t_s \in (-\infty, +\infty)$ $\bar{r}_s \in [0, \frac{1}{\omega})$
covers	(same as the static chart)	
Refs.		[13, rel. (2.7)]
metric		$ds^2 = \frac{1}{1+\omega^2\bar{r}_s^2}(dt^2 -$ $-\frac{d\bar{r}^2}{1+\omega^2\bar{r}_s^2} - \bar{r}_s^2 d\Omega_2^2)$
$\sqrt{\lvert g\rvert}$	$\frac{\sqrt{1-\omega^2\bar{\vec{x}}^2}}{(1+\omega^2\bar{\vec{x}}^2)^3}$	

Table 20: **Fermi (original de Sitter static) chart**

Introduced:	de Sitter, 1917 [42] [43]	
	cartesian $\{t_s, \vec{x}_F\}$	spherical $\{t_s, r_F, \theta, \phi\}$
def.	$Z^0 = \frac{\cos(\omega\lvert\vec{x}_F\rvert)}{\omega}\sinh(\omega t_s)$ $Z^i = \frac{\sin(\omega\lvert\vec{x}_F\rvert)}{\omega\lvert\vec{x}_F\rvert}x_F^i$ $Z^4 = \frac{\cos(\omega\lvert\vec{x}_F\rvert)}{\omega}\cosh(\omega t_s)$	$Z^0 = \frac{\cos(\omega r_F)}{\omega}\sinh(\omega t_s)$ $Z^1 = \frac{\sin(\omega r_F)}{\omega}\sin\theta\cos\phi$ $Z^2 = \frac{\sin(\omega r_F)}{\omega}\sin\theta\sin\phi$ $Z^3 = \frac{\sin(\omega r_F)}{\omega}\cos\theta$ $Z^4 = \frac{\cos(\omega r_F)}{\omega}\cosh(\omega t_s)$

Ranges of	$t_s \in (-\infty, +\infty)$	$t_s \in (-\infty, +\infty)$
coordinates	$\lvert\vec{x}_F\rvert \in \left[0, \frac{\pi}{2\omega}\right)$	$r_F \in \left[0, \frac{\pi}{2\omega}\right)$
covers	(same as the static chart)	
Refs.	[5, rel. (25,28,29)]	[5, rel. (31)] [14, rel. (1.1)] [13, rel. (2.5)]
metric	$ds^2 = \cos^2(\omega\lvert\vec{x}_F\rvert) - \left(\frac{x_{Fi}x_{Fj}}{\vec{x}_F^2} + \right.$ $\left. + \frac{\sin^2(\omega\lvert\vec{x}_F\rvert)}{\omega^2\lvert\vec{x}_F\rvert}(\delta_{ij} - \frac{x_{Fi}x_{Fj}}{\vec{x}_F^2})\right) dx_F^i dx_F^j$	$ds^2 = \cos^2(\omega r_F) dt_s^2 - dr_F^2$ $- \frac{\sin^2(\omega r_F)}{\omega^2} d\Omega_2^2$
$\sqrt{\lvert g\rvert}$	$\frac{\sin^2(\omega\lvert\vec{x}\rvert)\cos(\omega\lvert\vec{x}\rvert)}{\omega^2\vec{x}^2}$	

Table 21: **Tortoise (Regge- Wheeler static) chart**

	cartesian $\{t_s, \vec{x}^*\}$	spherical $\{t_s, r^*, \theta, \phi\}$
def.	$Z^0 = \frac{1}{\omega\cosh(\omega\lvert\vec{x}^*\rvert)}\cosh(\omega t_s)$ $Z^i = \frac{\tanh(\omega\lvert\vec{x}^*\rvert)}{\omega\lvert\vec{x}^*\rvert}x^{*i}$ $Z^4 = \frac{1}{\omega\cosh(\omega\lvert\vec{x}^*\rvert)}\omega\sinh(\omega t_s)$	$Z^0 = \frac{1}{\omega\cosh(\omega r^*)}\cosh(\omega t_s)$ $Z^1 = \frac{\tanh(\omega r^*)}{\omega r^*}\sin\theta\cos\phi$ $Z^2 = \frac{\tanh(\omega r^*)}{\omega r^*}\sin\theta\sin\phi$ $Z^3 = \frac{\tanh(\omega r^*)}{\omega r^*}\cos\theta$ $Z^4 = \frac{1}{\omega\cosh(\omega r^*)}\sinh(\omega t_s)$
Ranges of	$t_s \in (-\infty, +\infty)$	$t_s \in (-\infty, +\infty)$
coordinates	$\vec{x}^* \in \mathbb{R}^3_-$	$r^* \in (-\infty, 0)$
covers	(same as the static chart)	
Refs.		[1, rel. (18)]
metric		$ds^2 = \frac{1}{\cosh^2(\omega r^*)}(dt_s^2 - dr^{*2} -$ $- \frac{\sinh^2(\omega r^*)}{\omega^2} d\Omega_2^2)$

Table 22: **Static chart, isotropic coordinates**

	cartesian $\{t_s, \vec{\mathbf{x}}_s\}$	spherical $\{t_s, \rho_s, \theta, \phi\}$
def.	$Z^0 = \frac{1-\frac{\omega^2\vec{\mathbf{x}}_s^2}{4}}{1+\frac{\omega^2\vec{\mathbf{x}}_s^2}{4}}\sinh(\omega t_s)$ $Z^i = \frac{\vec{\mathbf{x}}_s^i}{1+\frac{\omega^2\vec{\mathbf{x}}_s^2}{4}}$ $Z^4 = \frac{1-\frac{\omega^2\vec{\mathbf{x}}_s^2}{4}}{1+\frac{\omega^2\vec{\mathbf{x}}_s^2}{4}}\cosh(\omega t_s)$	$Z^0 = \frac{1-\frac{\omega^2\rho_s^2}{4}}{1+\frac{\omega^2\rho_s^2}{4}}\sinh(\omega t_s)$ $Z^1 = \frac{\rho_s}{1+\frac{\omega^2\rho_s^2}{4}}\sin\theta\cos\phi$ $Z^2 = \frac{\rho_s}{1+\frac{\omega^2\rho_s^2}{4}}\sin\theta\sin\phi$ $Z^3 = \frac{\rho_s}{1+\frac{\omega^2\rho_s^2}{4}}\cos\theta$ $Z^4 = \frac{1-\frac{\omega^2\rho_s^2}{4}}{1+\frac{\omega^2\rho_s^2}{4}}\cosh(\omega t_s)$
Ranges of coordinates	$t_s \in (-\infty, +\infty)$ $\lvert\vec{\mathbf{x}}_s\rvert \in \left[0, \frac{2}{\omega}\right)$	$t_s \in (-\infty, +\infty)$ $\rho_s \in \left[0, \frac{2}{\omega}\right)$
covers	(same as the static chart)	
Refs.		[14, rel. (5.10-11)]
line elem.		$ds^2 = \frac{1}{\left(1+\frac{\omega^2\rho_s^2}{4}\right)^2}\left[\left(1-\frac{\omega^2\rho_s^2}{4}\right)^2 dt_s^2 - \right.$ $\left. -d\rho_s^2 - \rho_s^2 d\Omega_2^2\right]$

4 'Natural' charts

Table 23: **Minkowskian chart**

	cartesian $\{t_M, \vec{x}_s\}$	spherical $\{t_M, r_s, \theta, \phi\}$
def.	$Z^0 = t_M$ $Z^i = x_s^i$ $Z^4 = \frac{1}{\omega}\sqrt{1+\omega^2(t_M^2 - \vec{x}_s^2)}$	$Z^0 = t_M$ $Z^1 = r_s \sin\theta\cos\phi$ $Z^2 = r_s \sin\theta\sin\phi$ $Z^3 = r_s\cos\theta$ $Z^4 = \frac{1}{\omega}\sqrt{1+\omega^2(t_M^2 - r_s^2)}$
cover	half of the manifold [28]	
Refs.	[33, sec. 2.2]	
line elem.		$ds^2 = \frac{1}{1+\omega^2(t_M^2-r_s^2)}\Big((1-\omega^2 r_s^2)dt_M^2 +$ $+2\omega^2 t_M r_s dt_M dr_s - (1+\omega t_M^2)dr_s^2\Big) - r_s^2 d\Omega_2^2$
	$g_{\mu\nu} = \eta_{\mu\nu} - \frac{\omega^2 x_{s\mu}x_{s\nu}}{1+\omega^2(t_M^2-\vec{x}_s^2)}$ $g^{\mu\nu} = \eta^{\mu\nu} + \omega^2 x_s^\mu x_s^\nu$	
$\sqrt{\lvert g\rvert}$	$(1-\omega^2(t_M^2-\vec{x}_s^2))^{-\frac{1}{2}}$	
a tetrad	$\begin{cases} \hat{e}_\mu^{\hat{\alpha}} = \delta_\mu^\alpha + \frac{1}{t_M^2-\vec{x}_s^2}\cdot \\ \cdot\left(\sqrt{1+\omega^2(t_M^2-\vec{x}_s^2)}-1\right)\eta_{\mu\theta}x_M^\theta x_M^\alpha \end{cases}$ $\begin{cases} e_{\hat{\alpha}}^\mu = \delta_\alpha^\mu + \frac{1}{t_M^2-\vec{x}_s^2}\cdot \\ \cdot\left(\frac{1}{\sqrt{1+\omega^2(t_M^2-\vec{x}_s^2)}}-1\right)\eta_{\alpha\theta}x_M^\theta x_M^\mu \end{cases}$	
Killing vectors	$K_{\mu 4} = \frac{1}{\omega}\sqrt{1+\omega^2(t^2-\vec{x}^2)}\partial_\mu$ $K_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu$	

Table 24: **Beltrami chart**

	cartesian $\{t_B, \vec{x}_B\}$	spherical $\{t_B, r_B, \theta, \phi\}$
def.	$Z^0 = \frac{t_B}{\sqrt{1-\omega^2(t_B^2-\vec{x}_B^2)}}$ $Z^i = \frac{x_B^i}{\sqrt{1-\omega^2(t_B^2-\vec{x}_B^2)}}$ $Z^4 = \frac{1}{\omega\sqrt{1-\omega^2(t_B^2-\vec{x}_B^2)}}$	$Z^0 = \frac{t_B}{\sqrt{1-\omega^2(t_B^2-r_B^2)}}$ $Z^1 = \frac{r_B \sin\theta\cos\phi}{\sqrt{1-\omega^2(t_B^2-r_B^2)}}$ $Z^2 = \frac{r_B \sin\theta\sin\phi}{\sqrt{1-\omega^2(t_B^2-r_B^2)}}$ $Z^3 = \frac{r_B \cos\theta}{\sqrt{1-\omega^2(t_B^2-r_B^2)}}$ $Z^4 = \frac{1}{\omega\sqrt{1-\omega^2(t_B^2-r_B^2)}}$
cover	1/8 of the manifold [6]	
Refs.	[44] [6] [27] [33, sec. 2.4]	
	$g_{\mu\nu} = \frac{\left(1-\omega^2(t_B^2-\vec{x}_B^2)\right)\eta_{\mu\nu}+\omega^2 x_{B\mu}x_{B\nu}}{\left(1-\omega^2(t_B^2-\vec{x}_B^2)\right)^2}$ $g^{\mu\nu} = \left(1-\omega^2(t_B^2-\vec{x}_B^2)\right)\left(\eta^{\mu\nu}-\omega^2 x_B^\mu x_B^\nu\right)$	
$\sqrt{\lvert g\rvert}$	$(1-\omega^2(t_B^2-\vec{x}_B^2))^{-\frac{5}{2}}$	
a tetrad	$\begin{cases} \hat{e}_\mu^{\hat{\alpha}} = \left(1-\omega^2(t_B^2-\vec{x}_B^2)\right)\left[\delta_\mu^\alpha + \frac{1}{t_B^2-\vec{x}_B^2}\cdot \right. \\ \left. \cdot\left(\sqrt{1+\omega^2(t_B^2-\vec{x}_B^2)}-1\right)\eta_{\mu\theta}x_B^\theta x_B^\alpha\right] \end{cases}$ $\begin{cases} e_{\hat{\alpha}}^\mu = \frac{1}{1-\omega^2(t_B^2-\vec{x}_B^2)}\left[\delta_\alpha^\mu + \frac{1}{t_B^2-\vec{x}_B^2}\cdot \right. \\ \left. \cdot\left(\frac{1}{\sqrt{1+\omega^2(t_B^2-\vec{x}_B^2)}}-1\right)\eta_{\alpha\theta}x_B^\theta x_B^\mu\right] \end{cases}$	
Killing vectors	$K_{\mu 4} = \frac{1}{\omega}\partial_\mu - \omega x_\mu (x^\nu \partial_\nu)$ $K_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu$	

Table 25: **Stereographic (conformally Minkowski) chart**

	cartesian $\{t_{st}, \vec{x}_{st}\}$	spherical $\{t, r_{st}, \theta, \phi\}$
def.	$Z^0 = \dfrac{t_{st}}{1-\frac{\omega^2(t_{st}^2-\vec{x}_{st}^2)}{4}}$ $Z^i = \dfrac{x_{st}^i}{1-\frac{\omega^2(t_{st}^2-\vec{x}_{st}^2)}{4}}$ $Z^4 = \dfrac{1}{\omega}\left(1-\dfrac{2}{1-\frac{\omega^2(t_{st}^2-\vec{x}_{st}^2)}{4}}\right)$	$Z^0 = \dfrac{t_{st}}{1-\frac{\omega^2(t_{st}^2-\vec{x}_{st}^2)}{4}}$ $Z^1 = \dfrac{r_{st}\sin\theta\cos\phi}{1-\frac{\omega^2(t_{st}^2-r_{st}^2)}{4}}$ $Z^2 = \dfrac{r_{st}\sin\theta\sin\phi}{1-\frac{\omega^2(t_{st}^2-r_{st}^2)}{4}}$ $Z^3 = \dfrac{r_{st}\cos\theta}{1-\frac{\omega^2(t_{st}^2-r_{st}^2)}{4}}$ $Z^4 = \dfrac{1}{\omega}\left(1-\dfrac{2}{1-\frac{\omega^2(t_{st}^2-r_{st}^2)}{4}}\right)$
cover	3/4 of the manifold has cosmological horizon at $\lvert\vec{x}_{st}\rvert = r_{st} = \frac{2}{\omega}$	
Refs.	[33, sec. 2.3] [15, sec. A4] [35] [36]	
	$g_{\mu\nu} = \frac{\eta_{\mu\nu}}{1-\frac{\omega^2(t_{st}^2-\vec{x}_{st}^2)}{4}}$ $g^{\mu\nu} = \left(1-\frac{\omega^2(t_{st}^2-\vec{x}_{st}^2)}{4}\right)\eta^{\mu\nu}$	
$\sqrt{\lvert g\rvert}$	$\left(1-\frac{\omega^2(t_{st}^2-\vec{x}_{st}^2)}{4}\right)^{-4}$	
a tetrad	$\left\{\hat{e}_\mu^{\hat{\alpha}} = \sqrt{1-\dfrac{\omega^2(t_{st}^2-\vec{x}_{st}^2)}{4}}\,\delta_\mu^\alpha\right.$ $\left\{e_{\hat{\alpha}}^\mu = \dfrac{1}{\sqrt{1-\frac{\omega^2(t_{st}^2-\vec{x}_{st}^2)}{4}}}\delta_\alpha^\mu\right.$	
Killing vectors	$K_{\mu 4} = -\dfrac{1}{\omega}\left(1-\dfrac{\omega^2(t_{st}^2-\vec{x}_{st}^2)}{4}\right)\partial_\mu +$ $+\dfrac{\omega}{2}x_\mu(x^\nu\partial_\nu)$ $K_{\mu\nu} = x_\mu\partial_\nu - x_\nu\partial_\mu$	

5 'Hybrid' charts

Table 26: **de Sitter-Painlevé chart**

	cartesian $\{t, \vec{x}_s\}$	spherical $\{t, r_s, \theta, \phi\}$
def.	$Z^0 = \frac{1}{\omega}\sinh(\omega t) + \frac{\omega \vec{x}_s^2 e^{-\omega t}}{2}$ $Z^i = x_{si}$ $Z^4 = \frac{1}{\omega}\cosh(\omega t) - \frac{\omega \vec{x}_s^2 e^{-\omega t}}{2}$	$Z^0 = \frac{1}{\omega}\sinh(\omega t) + \frac{\omega r_s^2 e^{-\omega t}}{2}$ $Z^1 = r_s \sin\theta\cos\phi$ $Z^2 = r_s \sin\theta\sin\phi$ $Z^3 = r_s\cos\theta$ $Z^4 = \frac{1}{\omega}\cosh(\omega t) - \frac{\omega r_s^2 e^{-\omega t}}{2}$
cover	half of the manifold (same as spatially flat FLRW)	
	constant time slices are euclidean spaces	
	generator of t is a Killing vector	
Refs.		[4, rel. (2.3)]
		[2, rel. (4)]
line elem.	$ds^2 = (1-\omega^2\vec{x}_s^2)dt^2 +$	$ds^2 = (1-\omega^2 r_s^2)dt^2 +$
	$+2\omega\vec{x}_s d\vec{x}_s dt - d\vec{x}_s^2$	$+2\omega r_s dr_s dt - dr_s^2 - r_s^2 d\Omega_2^2$
metric	$\begin{cases} g_{00} = 1-\omega^2\vec{x}_s^2 \\ g_{0i} = g_{i0} = \omega x_s^i \\ g_{ij} = \eta_{ij} \end{cases}$	
	$\begin{cases} g^{00} = 1 \\ g^{0i} = g^{i0} = \omega x_s^i \\ g^{ij} = \eta^{ij} + \omega x_s^i x_s^j \end{cases}$	
$\sqrt{\lvert g \rvert}$	1	
1st type Christoffel coeff. (12 non-null)	$\Gamma_{0ii} = \omega$ $\Gamma_{0i0} = \Gamma_{00i} = -\omega^2 x_s^i$ $\Gamma_{i00} = \omega^2 x_s^i$	

2nd type Christoffel coeff. (40 non-null)

$$\Gamma^0_{00} = \omega^2 \vec{x}_s^2$$
$$\Gamma^0_{0i} = \Gamma^0_{i0} = -\omega^2 x_s^i$$
$$\Gamma^0_{ii} = \omega$$
$$\Gamma^i_{00} = -\omega^2 x_s^i (1 - \omega^2 \vec{x}_s^2)$$
$$\Gamma^i_{0j} = \Gamma^i_{j0} = -\omega^3 x_s^i x_s^j$$
$$\Gamma^i_{jj} = \omega^2 x_s^i$$

Killing vectors

$$K_{04} = \frac{1}{\omega}\partial_0$$
$$K_{0i} = -x_i e^{-\omega t}\partial_0 + \left(\frac{\sinh(\omega t)}{\omega} + \right.$$
$$\left. + \frac{\omega e^{-\omega t}\vec{x}^2}{2}\right)\partial_i$$
$$K_{i4} = x_i e^{-\omega t}\partial_0 + \left(\frac{\cosh(\omega t)}{\omega} - \right.$$
$$\left. - \frac{\omega e^{-\omega t}\vec{x}^2}{2}\right)\partial_i$$
$$K_{0i} + K_{i4} = \frac{e^{\omega t}}{\omega}\partial_i$$
$$K_{0i} - K_{i4} = -2e^{-\omega t} x_i \partial_0 -$$
$$- \frac{e^{\omega t}}{\omega}(1 - \omega^2 \vec{x}^2)\partial_i$$
$$K_{ij} = x_i \partial_j - x_j \partial_i$$

6 Anisotropic charts

Table 27: **Coordinates for Kantowski-Sachs and Bianchi III charts**

$t_s = r_{KS}$	$t_s = \frac{1}{2\omega}\ln\left\lvert\frac{\sinh(\omega r_{KS})+\tanh(\omega t_{B3})\cosh\Theta}{\sinh(\omega r_{KS})-\tanh(\omega t_{B3})\cosh\Theta}\right\rvert$
$\sinh(\omega t_{KS}) = \frac{1}{\sinh(\omega\bar{t}_{KS})}$	$\tanh(\omega t_{B3}) = \sin(\omega\bar{t}_{B3})$
$r_s = \frac{1}{\omega}\cosh(\omega t_{KS}) = \frac{1}{\omega}\coth(\omega t_{KS})$	$r_s = \sqrt{\sinh^2(\omega t_{B3})\sinh^2\Theta + \cosh^2(\omega t_{B3})\cos(\omega r_{B3})}$
	$\tan\theta = \frac{\sinh(\omega t_{B3})\sinh\Theta}{\cos(\omega r_{B3})}$

[14, p. 150]

Table 28: **Kantowski-Sachs charts**

	$\{t_{KS}, r_{KS}, \theta, \phi\}$	alternate $\{\bar{t}_{KS}, r_{KS}, \theta, \phi\}$
def.	$Z^0 = \dfrac{1}{\omega}\sinh(\omega t_{KS})\cosh(\omega r_{KS})$	$Z^0 = \dfrac{1}{\omega\sinh(\omega\bar{t}_{KS})}\cosh(\omega r_{KS})$
	$Z^1 = \dfrac{1}{\omega}\cosh(\omega t_{KS})\sin\theta\cos\phi$	$Z^1 = \dfrac{1}{\omega}\coth(\omega\bar{t}_{KS})\sin\theta\cos\phi$
	$Z^2 = \dfrac{1}{\omega}\cosh(\omega t_{KS})\sin\theta\sin\phi$	$Z^2 = \dfrac{1}{\omega}\coth(\omega\bar{t}_{KS})\sin\theta\sin\phi$
	$Z^3 = \dfrac{1}{\omega}\cosh(\omega t_{KS})\cos\theta$	$Z^3 = \dfrac{1}{\omega}\coth(\omega\bar{t}_{KS})\cos\theta$
	$Z^4 = \dfrac{1}{\omega}\sinh(\omega t_{KS})\sinh(\omega r_{KS})$	$Z^4 = \dfrac{1}{\omega\sinh(\omega\bar{t}_{KS})}\sinh(\omega r_{KS})$
cover	describe the part of the static de Sitter chart on the other side of the cosmological horizon at $r_{KS} = \frac{1}{\omega}$	
Refs.	[14, p. 149] [19, p. 53] [10] [9]	[14, p. 149]
line elem.	$ds^2 = dt_{KS}^2 - \sinh^2(\omega t_{KS})dr_{KS}^2 - \frac{\cosh^2(\omega t_{KS})}{\omega^2}d\Omega_2^2$	$ds^2 = \frac{1}{\sinh^2(\omega\bar{t}_{KS})}(d\bar{t}_{KS}^2 + dr_{KS}^2 - \frac{\cosh^2(\omega\bar{t}_{KS})}{\omega^2}d\Omega_2^2)$
$\sqrt{\lvert g\rvert}$	$\frac{\cosh^2(\omega t_{KS})\sinh(\omega t_{KS})\sin\theta}{\omega^3}$	$\frac{\cosh^2(\omega\bar{t}_{KS})\sin\theta}{\omega^3\sinh^3(\omega\bar{t}_{KS})}$

Table 29: **Bianchi III dS charts**

	$\{t_{B3}, r_{B3}, \Theta, \phi\}$	alternate $\{\bar{t}_{B3}, r_{B3}, \Theta, \phi\}$
def.	$Z^0 = \frac{1}{\omega}\sinh(\omega t_{B3})\cosh\Theta$ $Z^1 = \frac{1}{\omega}\sinh(\omega t_{B3})\sinh\Theta\cos\phi$ $Z^2 = \frac{1}{\omega}\sinh(\omega t_{B3})\sinh\Theta\sin\phi$ $Z^3 = \frac{1}{\omega}\cosh(\omega t_{B3})\cos(\omega r_{B3})$ $Z^4 = \frac{1}{\omega}\cosh(\omega t_{B3})\sin(\omega r_{B3})$	$Z^0 = \frac{1}{\omega}\tan(\omega\bar{t}_{B3})\cosh\Theta$ $Z^1 = \frac{1}{\omega}\tan(\omega\bar{t}_{B3})\sinh\Theta\cos\phi$ $Z^2 = \frac{1}{\omega}\tan(\omega\bar{t}_{B3})\sinh\Theta\sin\phi$ $Z^3 = \frac{1}{\omega\cos(\omega\bar{t}_{B3})}\cos(\omega r_{B3})$ $Z^4 = \frac{1}{\omega\cos(\omega\bar{t}_{B3})}\sin(\omega r_{B3})$
Refs.	[14, p. 150] [19, p. 53] [11]	[14, p. 150]
line elem.	$ds^2 = dt_{B3}^2 - \cosh^2(\omega t_{B3})dr_{B3}^2 -$ $-\frac{\sinh^2(\omega t_{B3})}{\omega^2}dH_2^2$	$ds^2 = \frac{1}{\cos(\omega\bar{t}_{B3})}(d\bar{t}_{B3}^2 -$ $-dr_{B3}^2 - \frac{\sin^2(\omega\bar{t}_{B3})}{\omega^2}dH_2^2)$
$\sqrt{\lvert g\rvert}$	$\frac{\cosh(\omega t_{B3})\sinh^2(\omega t_{B3})\sinh\Theta}{\omega^3}$	$\frac{\sin^2(\omega\bar{t}_{B3})\sinh\Theta}{\omega^3\cos^3(\omega\bar{t}_{B3})}$

Table 30: **Coordinates for conformally Rindler dS chart**

$$
\begin{aligned}
\xi &= \tfrac{\sqrt{1-\omega^2 r_s^2}}{\omega(1-\omega r_s \cos\theta)} & r &= \tfrac{1}{\omega}\tfrac{1+\omega^2(-\xi^2+y^2+z^2)}{1+\omega^2(\xi^2+y^2+z^2)} \\
y &= \tfrac{r_s}{1-\omega r_s \cos\theta}\sin\theta\cos\phi & \theta &= \arctan\tfrac{2\omega\sqrt{y^2+z^2}}{1-\omega^2(\xi^2+y^2+z^2)} \\
z &= \tfrac{r_s}{1-\omega r_s \cos\theta}\sin\theta\sin\phi & \phi &= \arctan\tfrac{z}{y}
\end{aligned}
$$

$$
\begin{aligned}
\sin\theta &= \tfrac{2\omega\sqrt{y^2+z^2}}{1+\omega^2(-\xi^2+y^2+z^2)} & \cos\theta &= \tfrac{1-\omega^2(\xi^2+y^2+z^2)}{1+\omega^2(-\xi^2+y^2+z^2)} \\
\sin\phi &= \tfrac{z}{\sqrt{y^2+z^2}} & \cos\phi &= \tfrac{y}{\sqrt{y^2+z^2}}
\end{aligned}
$$

Table 31: **Conformally Rindler dS chart**

	cartesian $\{t_s, \xi, y, z\}$
def.	$Z^0 = \dfrac{2\xi}{1+\omega^2(\xi^2+y^2+z^2)}\sinh(\omega t_s)$ $Z^1 = \dfrac{2y}{1+\omega^2(\xi^2+y^2+z^2)}$ $Z^2 = \dfrac{2z}{1+\omega^2(\xi^2+y^2+z^2)}$ $Z^3 = \dfrac{1-\omega^2(\xi^2+y^2+z^2)}{1+\omega^2(\xi^2+y^2+z^2)}$ $Z^4 = \dfrac{2\xi}{1+\omega^2(\xi^2+y^2+z^2)}\cosh(\omega t_s)$
Refs.	[17, p. 187] [7]
line elem.	$ds^2 = \frac{4}{1+\omega^2(\xi^2+y^2+z^2)}(\omega^2\xi^2 dt_s^2 - d\xi^2 - dy^2 - dz^2)$
$\sqrt{\lvert g\rvert}$	$\frac{16\omega\xi}{(1+\omega^2(\xi^2+y^2+z^2))^4}$

7 Charts with null coordinates

Table 32: **Other coordinates on de Sitter spacetime**

Coordinate	Symbol	Relations	References
Static time	t_s	$t_s = \frac{u+v}{2}$	-
	[1, 20, t] [14, T]	$t_s = \frac{1}{\omega} \operatorname{arctanh} \frac{\tilde{V}}{\tilde{U}}$	[1, from (7)]
Static coordinate	r_s	$r_s = \frac{1}{\omega} \frac{1-\omega^2(\tilde{U}^2+\tilde{V}^2)}{1+\omega^2(\tilde{U}^2-\tilde{V}^2)}$	[1, from (6)]
		$r_s = \frac{1}{\omega} \frac{1+\omega^2 UV}{1-\omega^2 UV}$	[20, rel. (23)]
Tortoise coordinate	r^*	$r^* = \frac{v-u}{2}$	-
Outgoing (retarded) null	u	$u = t_s - r^*$	[1, rel. (19)]
(lightcone) Eddington-	[14]	$u = t_s - \frac{1}{2\omega} \ln \frac{1+\omega r_s}{1-\omega r_s}$	[20, rel. (20)]
Finkelstein coordinates	[20, x^-] [1, U]	$u = \frac{1}{\omega} \ln(\omega U)$	[20, rel. (22a)]
Ingoing (advanced) null	v	$v = t_s + r^*$	[1, rel. (20)]
(lightcone) Eddington-	[14]	$v = t_s + \frac{1}{2\omega} \ln \frac{1+\omega r_s}{1-\omega r_s}$	[20, rel. (18)]
Finkelstein coordinates	[20, x^+] [1, V]	$v = -\frac{1}{\omega} \ln(-\omega V)$	[20, rel. (22b)]
Kruskal-like	U	$U = \frac{1}{\omega} e^{\omega u}$	[20, from (22a)]
coordinates	[20] [31] [14, $\hat{U}$]	$U = \frac{\tilde{U}+\tilde{V}}{2}$	-
(lightcone variant)	V	$V = -\frac{1}{\omega} e^{-\omega v}$	[20, from (22b)]
	[20] [31] [14, $\hat{V}$]	$V = \frac{\tilde{U}-\tilde{V}}{2}$	-
Kruskal-like	$\tilde{U}$	$\tilde{U} = \frac{1}{\omega} e^{-\omega r^*} \cosh(\omega t_s)$	[1, rel. (21)]
coordinates	[1, u] [14, U]	$\tilde{U} = \frac{e^{\omega u}+e^{-\omega v}}{2\omega}$	[1, rel. (23)]
		$\tilde{U} = \frac{U-V}{2}$	-
	$\tilde{V}$	$\tilde{V} = \frac{1}{\omega} e^{-\omega r^*} \sinh(\omega t_s)$	[1, rel. (24)]
	[1] [14, V]	$\tilde{V} = \frac{e^{\omega u}-e^{-\omega v}}{2\omega}$	[1, rel. (22)]
		$\tilde{V} = \frac{U+V}{2}$	-

	$\bar{V}$	$\bar{V} = \frac{\omega^2}{V}$	[8] [14]

Table 33: **Eddington-Finkelstein charts**

	ingoing $\{v, r_s, \theta, \phi\}$	outgoing $\{u, r_s, \theta, \phi\}$
def.	$Z^0 = \frac{1}{\omega}\sinh(\omega v) - r_s\cosh(\omega r_s)$ $Z^1 = r_s \sin\theta\cos\phi$ $Z^2 = r_s\sin\theta\sin\phi$ $Z^3 = r_s\cos\theta$ $Z^4 = \frac{1}{\omega}\cosh(\omega v) - r_s\sinh(\omega r_s)$	$Z^0 = \frac{1}{\omega}\sinh(\omega u) + r_s\cosh(\omega r_s)$ $Z^1 = r_s \sin\theta\cos\phi$ $Z^2 = r_s\sin\theta\sin\phi$ $Z^3 = r_s\cos\theta$ $Z^4 = \frac{1}{\omega}\cosh(\omega u) + r_s\sinh(\omega r_s)$
Refs.	[20, rel.(19), missing dr_s^2]	
metric	$ds^2 = -(1-\omega^2 r_s^2)dv^2 + 2dvdr_s -$ $-2dr_s^2 - r_s^2 d\Omega_2^2$	$ds^2 = -(1-\omega^2 r_s^2)du^2 - 2dudr_s -$ $-2dr_s^2 - r_s^2 d\Omega_2^2$

Table 34: **Kruskal-like charts**

	$\{u, v, \theta, \phi\}$	$\{U, V, \theta, \phi\}$
introduced:		Gibbons, Hawking (1977) [31]
def.	$Z^0 = \frac{1}{\omega}\frac{\sinh\frac{\omega(v+u)}{2}}{\cosh\frac{\omega(v-u)}{2}}$ $Z^1 = \frac{1}{\omega}\tanh\frac{\omega(v-u)}{2}\sin\theta\cos\phi$ $Z^2 = \frac{1}{\omega}\tanh\frac{\omega(v-u)}{2}\sin\theta\sin\phi$ $Z^3 = \frac{1}{\omega}\tanh\frac{\omega(v-u)}{2}\cos\theta$ $Z^4 = \frac{1}{\omega}\frac{\cosh\frac{\omega(v+u)}{2}}{\cosh\frac{\omega(v-u)}{2}}$	$Z^0 = \frac{U+V}{\omega^2 UV - 1}$ $Z^1 = -\frac{1}{\omega}\frac{\omega^2 UV + 1}{\omega^2 UV - 1}\sin\theta\cos\phi$ $Z^2 = -\frac{1}{\omega}\frac{\omega^2 UV + 1}{\omega^2 UV - 1}\sin\theta\sin\phi$ $Z^3 = -\frac{1}{\omega}\frac{\omega^2 UV + 1}{\omega^2 UV - 1}\cos\theta$ $Z^4 = \frac{U-V}{\omega^2 UV - 1}$
metric	$ds^2 = -\frac{dudv}{\cosh^2\frac{\omega(v-u)}{2}} + \tanh^2\frac{\omega(v-u)}{2}d\Omega_2^2$	$ds^2 = \frac{2dUdV}{(\omega^2 UV-1)^2} - \frac{1}{\omega^2}\frac{(\omega^2 UV+1)^2}{(\omega^2 UV-1)^2}d\Omega_2^2$

Refs.	[20, rel.(21)] [14, rel. (3.8)]	[20, rel.(24)]

	$\{\tilde{U}, \tilde{V}, \theta, \phi\}$	$\{U, \bar{V}, \theta, \phi\}$
introduced:	Beck, Inomata (1984) [1]	Torrence, Couch (1986) [8]
studied:	Blau, Guendelman, Guth (1987) [32]	
def.	$Z^0 = \frac{\tilde{U}}{\frac{\omega^2(\tilde{U}^2-\tilde{V}^2)}{4}-1}$ $Z^1 = -\frac{1}{\omega}\frac{\frac{\omega^2(\tilde{U}^2-\tilde{V}^2)}{4}+1}{\frac{\omega^2(\tilde{U}^2-\tilde{V}^2)}{4}-1}\sin\theta\cos\phi$ $Z^2 = -\frac{1}{\omega}\frac{\frac{\omega^2(\tilde{U}^2-\tilde{V}^2)}{4}+1}{\frac{\omega^2(\tilde{U}^2-\tilde{V}^2)}{4}-1}\sin\theta\sin\phi$ $Z^3 = -\frac{1}{\omega}\frac{\frac{\omega^2(\tilde{U}^2-\tilde{V}^2)}{4}+1}{\frac{\omega^2(\tilde{U}^2-\tilde{V}^2)}{4}-1}\cos\theta$ $Z^4 = \frac{\tilde{V}}{\frac{\omega^2(\tilde{U}^2-\tilde{V}^2)}{4}-1}$	$Z^0 = \frac{U\bar{V}+\frac{1}{\omega^2}}{U-\bar{V}}$ $Z^1 = \frac{1}{\omega}\frac{U+\bar{V}}{U-\bar{V}}\sin\theta\cos\phi$ $Z^2 = \frac{1}{\omega}\frac{U+\bar{V}}{U-\bar{V}}\sin\theta\sin\phi$ $Z^3 = \frac{1}{\omega}\frac{U+\bar{V}}{U-\bar{V}}\cos\theta$ $Z^4 = \frac{U\bar{V}+\frac{1}{\omega^2}}{U-\bar{V}}$
metric	$ds^2 = \frac{1}{\frac{\omega^2(\tilde{U}^2-\tilde{V}^2)}{4}+1}\Big[d\tilde{U}^2 - d\tilde{V}^2 - $ $- \left(\frac{\omega^2(\tilde{U}^2-\tilde{V}^2)}{4}-1\right)d\Omega_2^2\Big]$	$ds^2 = -\frac{1}{\omega^2(U-\bar{V})^2}(4dUd\bar{V} -$ $-(U+\bar{V})^2 d\Omega_2^2)$
Refs.	[1, rel. (8-9)] [14, rel. (3.12)]	[14, rel. (3.13)]

Table 35: **Constant curvature coordinates**

	$\{\hat{U}, \hat{V}, \hat{\xi}, \bar{\hat{\xi}}\}$

def.	$Z^0 = \frac{1}{\sqrt{2}} \frac{\hat{U}+\hat{V}}{1-\frac{\omega^2}{2}(\hat{U}\hat{V}-\hat{\xi}\bar{\hat{\xi}})}$ $Z^1 = \frac{1}{\sqrt{2}} \frac{\hat{V}-\hat{U}}{1-\frac{\omega^2}{2}(\hat{U}\hat{V}-\hat{\xi}\bar{\hat{\xi}})}$ $Z^2 = \frac{1}{\sqrt{2}} \frac{\hat{\xi}+\bar{\hat{x}}i}{1-\frac{\omega^2}{2}(\hat{U}\hat{V}-\hat{\xi}\bar{\hat{\xi}})}$ $Z^3 = -\frac{i}{\sqrt{2}} \frac{\hat{\xi}-\bar{\hat{x}}i}{1-\frac{\omega^2}{2}(\hat{U}\hat{V}-\hat{\xi}\bar{\hat{\xi}})}$ $Z^4 = \frac{1}{\omega} \frac{1+\frac{\omega^2}{2}(\hat{U}\hat{V}-\hat{\xi}\bar{\hat{\xi}})}{1-\frac{\omega^2}{2}(\hat{U}\hat{V}-\hat{\xi}\bar{\hat{\xi}})}$
ranges	$\hat{U}, \hat{V} \in \mathbb{R}$, $\hat{\xi} \in \mathbb{C}$
cover	entire manifold, except singular surface where $\hat{U} = \infty$, $\hat{V} = \infty$
Refs.	[19, p. 51-52]
line elem.	$ds^2 = \frac{-2d\hat{U}\hat{V}+2\hat{\xi}\bar{\hat{\xi}}}{\left(1-\frac{\omega^2}{2}(\hat{U}\hat{V}-\hat{\xi}\bar{\hat{\xi}})\right)^2}$
$\sqrt{\lvert g \rvert}$	$\frac{1}{\left(1-\frac{\omega^2}{2}(\hat{U}\hat{V}-\hat{\xi}\bar{\hat{\xi}})\right)^4}$

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